

Exercise 7

Use residues to find the Cauchy principal values of the integrals in Exercises 6 and 7.

$$\int_{-\infty}^{\infty} \frac{x \, dx}{(x^2 + 1)(x^2 + 2x + 2)}.$$

Ans. $-\pi/5$.

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z}{(z^2 + 1)(z^2 + 2z + 2)},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} (z^2 + 1)(z^2 + 2z + 2) &= 0 \\ z^2 + 1 = 0 \quad \text{or} \quad z^2 + 2z + 2 &= 0 \\ z = \pm i \quad \text{or} \quad z = \frac{-2 \pm \sqrt{4 - 8}}{2} \\ & z = -1 + i \quad \text{or} \quad z = -1 - i \end{aligned}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = i$ and $z = -1 + i$.

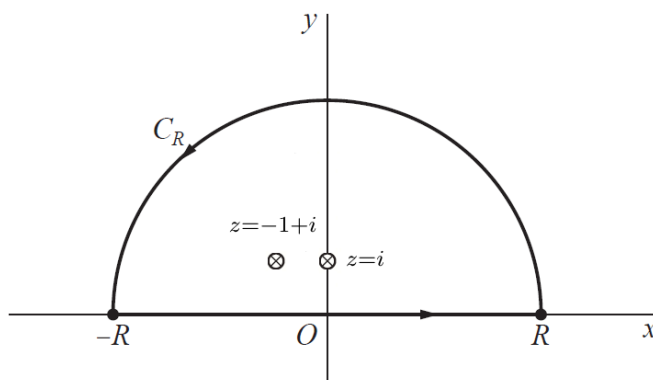


Figure 1: This is Fig. 93 with the singularity at $z = -1 + i$ marked.

According to Cauchy's residue theorem, the integral of $z/[(z^2 + 1)(z^2 + 2z + 2)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z \, dz}{(z^2 + 1)(z^2 + 2z + 2)} = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} + \operatorname{Res}_{z=-1+i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} \right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\begin{aligned} \int_L \frac{z \, dz}{(z^2 + 1)(z^2 + 2z + 2)} + \int_{C_R} \frac{z \, dz}{(z^2 + 1)(z^2 + 2z + 2)} \\ = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} + \operatorname{Res}_{z=-1+i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} \right] \end{aligned}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: \quad z &= r, & r &= -R \rightarrow r = R \\ C_R: \quad z &= Re^{i\theta}, & \theta &= 0 \rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\begin{aligned} \int_{-R}^R \frac{r \, dr}{(r^2 + 1)(r^2 + 2r + 2)} + \int_{C_R} \frac{z \, dz}{(z^2 + 1)(z^2 + 2z + 2)} \\ = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} + \operatorname{Res}_{z=-1+i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} \right]. \end{aligned}$$

Take the limit now as $R \rightarrow \infty$.

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{r \, dr}{(r^2 + 1)(r^2 + 2r + 2)} + \lim_{R \rightarrow \infty} \int_{C_R} \frac{z \, dz}{(z^2 + 1)(z^2 + 2z + 2)} \\ = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} + \operatorname{Res}_{z=-1+i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} \right] \end{aligned}$$

If the first limit exists, then it's defined to be the Cauchy principal value (P.V.) of the integral. The second limit is zero. Proof for this statement will be given at the end.

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{r \, dr}{(r^2 + 1)(r^2 + 2r + 2)} = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} + \operatorname{Res}_{z=-1+i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} \right]$$

The denominator can be written as $(z^2 + 1)(z^2 + 2z + 2) = (z + i)(z - i)[z - (-1 + i)][z - (-1 - i)]$. From this we see that the multiplicities of the $z - i$ and $z - (-1 + i)$ factors are both 1. The residues at $z = i$ and $z = -1 + i$ can then be calculated by

$$\begin{aligned} \operatorname{Res}_{z=i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} &= \phi_1(i) \\ \operatorname{Res}_{z=-1+i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} &= \phi_2(-1 + i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z - i$ and $z - (-1 + i)$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{z}{(z + i)[z - (-1 + i)][z - (-1 - i)]} \Rightarrow \phi_1(i) = \frac{1}{10} - \frac{i}{5} \\ \phi_2(z) &= \frac{z}{(z + i)(z - i)[z - (-1 - i)]} \Rightarrow \phi_2(-1 + i) = -\frac{1}{10} + \frac{3i}{10} \end{aligned}$$

So then

$$\begin{aligned} \operatorname{Res}_{z=i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} &= \frac{1}{10} - \frac{i}{5} \\ \operatorname{Res}_{z=-1+i} \frac{z}{(z^2 + 1)(z^2 + 2z + 2)} &= -\frac{1}{10} + \frac{3i}{10} \end{aligned}$$

and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{r \, dr}{(r^2 + 1)(r^2 + 2r + 2)} = 2\pi i \left(\frac{1}{10} - \frac{i}{5} - \frac{1}{10} + \frac{3i}{10} \right) = 2\pi i \left(\frac{i}{10} \right) = -\frac{\pi}{5}.$$

Therefore, changing the dummy integration variable to x ,

$$\boxed{\text{P.V.} \int_{-\infty}^{\infty} \frac{x \, dx}{(x^2 + 1)(x^2 + 2x + 2)} = -\frac{\pi}{5}.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \rightarrow \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π .

$$\begin{aligned} \int_{C_R} \frac{z dz}{(z^2 + 1)(z^2 + 2z + 2)} &= \int_{C_R} \frac{z dz}{(z^2 + 1)[z - (-1 + i)][z - (-1 - i)]} \\ &= \int_0^\pi \frac{Re^{i\theta}(Rie^{i\theta} d\theta)}{[(Re^{i\theta})^2 + 1][Re^{i\theta} - (-1 + i)][Re^{i\theta} - (-1 - i)]} \\ &= \int_0^\pi \frac{R^2 ie^{i2\theta} d\theta}{(R^2 e^{i2\theta} + 1)[Re^{i\theta} + (1 - i)][Re^{i\theta} + (1 + i)]} \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{z dz}{(z^2 + 1)(z^2 + 2z + 2)} \right| &= \left| \int_0^\pi \frac{R^2 ie^{i2\theta} d\theta}{(R^2 e^{i2\theta} + 1)[Re^{i\theta} + (1 - i)][Re^{i\theta} + (1 + i)]} \right| \\ &\leq \int_0^\pi \left| \frac{R^2 ie^{i2\theta}}{(R^2 e^{i2\theta} + 1)[Re^{i\theta} + (1 - i)][Re^{i\theta} + (1 + i)]} \right| d\theta \\ &= \int_0^\pi \frac{|R^2 ie^{i2\theta}|}{|R^2 e^{i2\theta} + 1| |Re^{i\theta} + (1 - i)| |Re^{i\theta} + (1 + i)|} d\theta \\ &= \int_0^\pi \frac{R^2}{|R^2 e^{i2\theta} + 1| |Re^{i\theta} + (1 - i)| |Re^{i\theta} + (1 + i)|} d\theta \\ &\leq \int_0^\pi \frac{R^2}{(|R^2 e^{i2\theta}| - |1|)(|Re^{i\theta}| - |1 - i|)(|Re^{i\theta}| - |1 + i|)} d\theta \\ &= \int_0^\pi \frac{R^2}{(R^2 - 1)(R - \sqrt{2})^2} d\theta \\ &= \pi \frac{R^2}{(R^2 - 1)(R - \sqrt{2})^2} \end{aligned}$$

Now take the limit of both sides as $R \rightarrow \infty$.

$$\begin{aligned} \lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z dz}{(z^2 + 1)(z^2 + 2z + 2)} \right| &\leq \lim_{R \rightarrow \infty} \pi \frac{R^2}{(R^2 - 1)(R - \sqrt{2})^2} \\ &= \lim_{R \rightarrow \infty} \frac{\pi}{R^2} \frac{1}{\left(1 - \frac{1}{R^2}\right) \left(1 - \frac{\sqrt{2}}{R}\right)^2} \end{aligned}$$

The limit on the right side is zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z dz}{(z^2 + 1)(z^2 + 2z + 2)} \right| \leq 0$$

The magnitude of a number cannot be negative.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z dz}{(z^2 + 1)(z^2 + 2z + 2)} \right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \rightarrow \infty} \int_{C_R} \frac{z dz}{(z^2 + 1)(z^2 + 2z + 2)} = 0.$$