# Exercise 12

Follow the steps below to evaluate the *Fresnel integrals*, which are important in diffraction theory:

$$\int_0^\infty \cos(x^2)\,dx = \int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

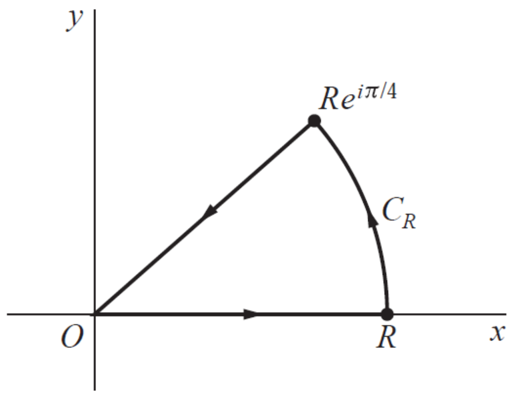


**FIGURE 99**

(a) By integrating the function $\exp(iz^2)$ around the positively oriented boundary of the sector $0 \le r \le R$, $0 \le \theta \le \pi/4$ (Fig. 99) and appealing to the Cauchy-Goursat theorem, show that

$$\int_0^R \cos(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Re}\int_{C_R} e^{iz^2}\,dz$$

and

$$\int_0^R \sin(x^2)\,dx = \frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \operatorname{Im}\int_{C_R} e^{iz^2}\,dz,$$

where $C_R$ is the arc $z = Re^{i\theta}$ $(0 \le \theta \le \pi/4)$.

(b) Show that the value of the integral along the arc $C_R$ in part $(a)$ tends to zero as $R$ tends to infinity by obtaining the inequality

$$\left|\int_{C_R} e^{iz^2}\,dz\right| \le \frac{R}{2}\int_0^{\pi/2} e^{-R^2\sin\phi}\,d\phi$$

and then referring to the form (2), Sec. 81, of Jordan's inequality.

(c) Use the results in parts $(a)$ and $(b)$, together with the known integration formula*

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2},$$

to complete the exercise.

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*The usual way to evaluate this integral is by writing its square as

$$\int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

and then evaluating this iterated integral by changing to polar coordinates. Details are given in, for example, A. E. Taylor and W. R. Mann, "Advanced Calculus," 3d ed., pp. 680–681, 1983.