Exercise 3

Use residues to evaluate the improper integrals in Exercises 1 through 8.

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx \quad (a>0,\ b>0).$$

$$Ans.\ \frac{\pi}{4b^3}(1+ab)e^{-ab}.$$

Solution

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty,\infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \int_{-\infty}^\infty \frac{\cos ax}{2(x^2+b^2)^2}\,dx$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{e^{iaz}}{2(z^2+b^2)^2},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$(z^2+b^2)^2 = 0$$
$$z^2+b^2 = 0$$
$$z = \pm ib$$

The singular point of interest to us is the one that lies within the closed contour, $z = ib$.

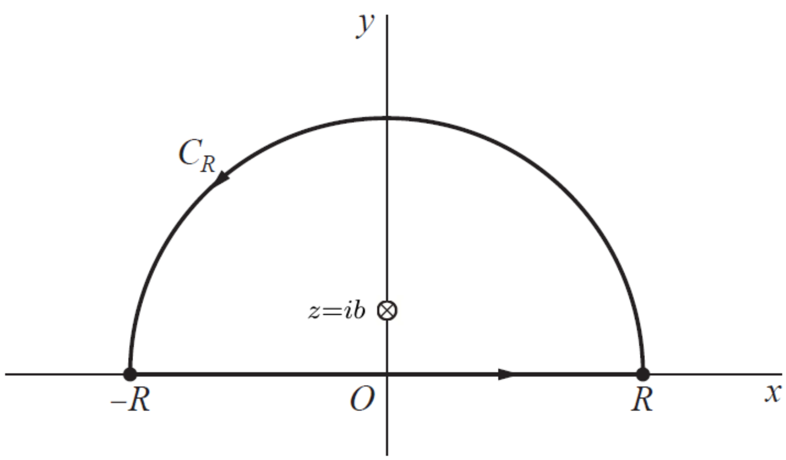

Figure 1: This is Fig. 93 with the singularity at $z = ib$ marked.

According to Cauchy's residue theorem, the integral of $e^{iaz}/[2(z^2+b^2)^2]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz = 2\pi i \operatorname*{Res}_{z=ib} \frac{e^{iaz}}{2(z^2+b^2)^2}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz + \int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz = 2\pi i \operatorname*{Res}_{z=ib} \frac{e^{iaz}}{2(z^2+b^2)^2}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L:&\quad z = r, & r=-R \quad &\to \quad r=R \\ C_R:&\quad z = Re^{i\theta}, & \theta=0 \quad &\to \quad \theta=\pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{e^{iar}}{2(r^2+b^2)^2}\,dr + \int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz = 2\pi i \operatorname*{Res}_{z=ib} \frac{e^{iaz}}{2(z^2+b^2)^2}.$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{e^{iar}}{2(r^2+b^2)^2}\,dr = 2\pi i \operatorname*{Res}_{z=ib} \frac{e^{iaz}}{2(z^2+b^2)^2}.$$

The denominator can be written as $2(z^2+b^2)^2 = 2(z+ib)^2(z-ib)^2$. From this we see that the multiplicity of the $z-ib$ factor is 2. The residue at $z=ib$ can then be calculated by

$$\operatorname*{Res}_{z=ib} \frac{e^{iaz}}{2(z^2+b^2)^2} = \frac{\phi^{(2-1)}(ib)}{(2-1)!} = \phi'(ib),$$

where $\phi(z)$ is equal to $f(z)$ without the $z-ib$ factors.

$$\phi(z) = \frac{e^{iaz}}{2(z+ib)^2}$$

Take a derivative of it to find $\phi'(z)$.

$$\phi'(z) = \frac{e^{iaz}(iaz-ab-2)}{2(z+ib)^3} \quad \Rightarrow \quad \phi'(ib) = \frac{e^{ia(ib)}[ia(ib)-ab-2]}{2(2ib)^3} = \frac{e^{-ab}(-2ab-2)}{16i^3b^3} = \frac{e^{-ab}(ab+1)}{8ib^3}$$

So then

$$\operatorname*{Res}_{z=ib} \frac{e^{iaz}}{2(z^2+b^2)^2} = \frac{e^{-ab}(1+ab)}{8ib^3}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{iar}}{2(r^2+b^2)^2}\,dr &= 2\pi i \left[\frac{e^{-ab}(1+ab)}{8ib^3}\right] \\ \int_{-\infty}^{\infty} \frac{\cos ar + i\sin ar}{2(r^2+b^2)^2}\,dr &= \frac{\pi}{4b^3}(1+ab)e^{-ab} \\ \int_{-\infty}^{\infty} \frac{\cos ar}{2(r^2+b^2)^2}\,dr + i\int_{-\infty}^{\infty} \frac{\sin ar}{2(r^2+b^2)^2}\,dr &= \frac{\pi}{4b^3}(1+ab)e^{-ab}. \end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{\cos ar}{2(r^2+b^2)^2}\,dr = \frac{\pi}{4b^3}(1+ab)e^{-ab} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin ar}{2(r^2+b^2)^2}\,dr = 0$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab}.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}
\int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz &= \int_0^\pi \frac{e^{iaRe^{i\theta}}}{2[(Re^{i\theta})^2+b^2]^2}(Rie^{i\theta}\,d\theta) \\
&= \int_0^\pi \frac{e^{iaR(\cos\theta+i\sin\theta)}}{(R^2e^{i2\theta}+b^2)^2}\left(\frac{Rie^{i\theta}}{2}\,d\theta\right) \\
&= \int_0^\pi \frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{(R^2e^{i2\theta}+b^2)^2}\left(\frac{Rie^{i\theta}}{2}\,d\theta\right)
\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz\right| &= \left|\int_0^\pi \frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{(R^2e^{i2\theta}+b^2)^2}\left(\frac{Rie^{i\theta}}{2}\,d\theta\right)\right| \\
&\le \int_0^\pi \left|\frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{(R^2e^{i2\theta}+b^2)^2}\left(\frac{Rie^{i\theta}}{2}\right)\right| d\theta \\
&= \int_0^\pi \frac{\left|e^{iaR\cos\theta}\right|\left|e^{-aR\sin\theta}\right|}{|(R^2e^{i2\theta}+b^2)^2|}\left|\frac{Rie^{i\theta}}{2}\right| d\theta \\
&= \int_0^\pi \frac{e^{-aR\sin\theta}}{|R^2e^{i2\theta}+b^2|^2}\frac{R}{2}\,d\theta \\
&\le \int_0^\pi \frac{e^{-aR\sin\theta}}{(|R^2e^{i2\theta}|-|b^2|)^2}\frac{R}{2}\,d\theta \\
&= \int_0^\pi \frac{e^{-aR\sin\theta}}{(R^2-b^2)^2}\frac{R}{2}\,d\theta \\
&= \int_0^\pi \frac{e^{-aR\sin\theta}}{\left(1-\frac{b^2}{R^2}\right)^2}\frac{d\theta}{2R^3}
\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz\right| \le \lim_{R\to\infty}\int_0^\pi \frac{e^{-aR\sin\theta}}{\left(1-\frac{b^2}{R^2}\right)^2}\frac{d\theta}{2R^3}$$

Because the limits of integration do not depend on R, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz\right| \le \int_0^\pi \lim_{R\to\infty}\frac{e^{-aR\sin\theta}}{\left(1-\frac{b^2}{R^2}\right)^2}\frac{d\theta}{2R^3}$$

Since θ lies between 0 and π, the sine of θ is positive. a is also positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz\right| \le 0$$

The magnitude of a number cannot be negative, and only zero has a magnitude of zero. Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz\right| = 0 \quad \to \quad \lim_{R\to\infty}\int_{C_R} \frac{e^{iaz}}{2(z^2+b^2)^2}\,dz = 0.$$