Exercise 5

Use residues to evaluate the improper integrals in Exercises 1 through 8.

$$\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4 + 4}\, dx \quad (a > 0).$$

$$Ans.\ \frac{\pi}{2} e^{-a} \sin a.$$

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z e^{iaz}}{z^4 + 4},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$z^4 + 4 = 0$$

$$z = \sqrt[4]{4} \exp\left[i\left(\frac{\pi + 2k\pi}{4}\right)\right], \quad k = 0, 1, 2, 3 \quad \rightarrow \quad \begin{cases} z_1 = \sqrt{2} e^{i\pi/4} = 1 + i \\ z_2 = \sqrt{2} e^{i3\pi/4} = -1 + i \\ z_3 = \sqrt{2} e^{i5\pi/4} = -1 - i \\ z_4 = \sqrt{2} e^{i7\pi/4} = 1 - i \end{cases}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = z_1$ and $z = z_2$.

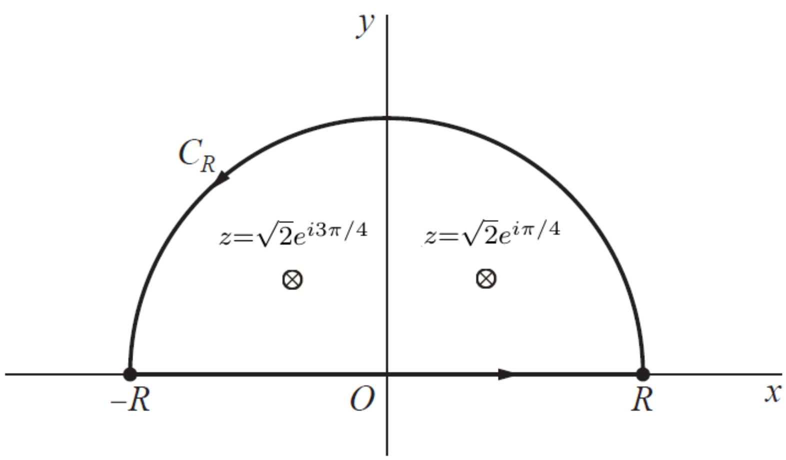

Figure 1: This is Fig. 93 with the singularities at $z = z_1$ and $z = z_2$ marked.

According to Cauchy's residue theorem, the integral of $ze^{iaz}/(z^4 + 4)$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z e^{iaz}}{z^4 + 4}\, dz = 2\pi i \left(\operatorname*{Res}_{z=z_1} \frac{z e^{iaz}}{z^4 + 4} + \operatorname*{Res}_{z=z_2} \frac{z e^{iaz}}{z^4 + 4} \right)$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{z e^{iaz}}{z^4 + 4}\, dz + \int_{C_R} \frac{z e^{iaz}}{z^4 + 4}\, dz = 2\pi i \left(\operatorname*{Res}_{z=z_1} \frac{z e^{iaz}}{z^4 + 4} + \operatorname*{Res}_{z=z_2} \frac{z e^{iaz}}{z^4 + 4} \right)$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, && r = -R \quad \to \quad r = R \\ C_R: &\quad z = Re^{i\theta}, && \theta = 0 \quad \to \quad \theta = \pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{re^{iar}}{r^4+4}\,dr + \int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz = 2\pi i \left(\operatorname*{Res}_{z=z_1} \frac{ze^{iaz}}{z^4+4} + \operatorname*{Res}_{z=z_2} \frac{ze^{iaz}}{z^4+4} \right).$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{re^{iar}}{r^4+4}\,dr = 2\pi i \left(\operatorname*{Res}_{z=z_1} \frac{ze^{iaz}}{z^4+4} + \operatorname*{Res}_{z=z_2} \frac{ze^{iaz}}{z^4+4} \right)$$

The denominator can be written as $z^4+4 = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$. From this we see that the multiplicities of the $z-z_1$ and $z-z_2$ factors are both 1. The residues at $z=z_1$ and $z=z_2$ can then be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=z_1} \frac{ze^{iaz}}{z^4+4} &= \phi_1(z_1) \\ \operatorname*{Res}_{z=z_2} \frac{ze^{iaz}}{z^4+4} &= \phi_2(z_2), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z-z_1$ and $z-z_2$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{ze^{iaz}}{(z-z_2)(z-z_3)(z-z_4)} \quad \Rightarrow \quad \phi_1(z_1) = \frac{1}{8i}e^{(-1+i)a} \\ \phi_2(z) &= \frac{ze^{iaz}}{(z-z_1)(z-z_3)(z-z_4)} \quad \Rightarrow \quad \phi_2(z_2) = -\frac{1}{8i}e^{(-1-i)a} \end{aligned}$$

So then

$$\begin{aligned} \operatorname*{Res}_{z=z_1} \frac{ze^{iaz}}{z^4+4} &= \frac{1}{8i}e^{(-1+i)a} \\ \operatorname*{Res}_{z=z_2} \frac{ze^{iaz}}{z^4+4} &= -\frac{1}{8i}e^{(-1-i)a} \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{re^{iar}}{r^4+4}\,dr &= 2\pi i \left[\frac{1}{8i}e^{(-1+i)a} - \frac{1}{8i}e^{(-1-i)a} \right] \\ \int_{-\infty}^{\infty} \frac{r\cos ar + ir\sin ar}{r^4+4}\,dr &= \frac{\pi i}{2}e^{-a}\left(\frac{e^{ia}-e^{-ia}}{2i} \right) \\ \int_{-\infty}^{\infty} \frac{r\cos ar}{r^4+4}\,dr + i\int_{-\infty}^{\infty} \frac{r\sin ar}{r^4+4}\,dr &= \frac{i\pi}{2}e^{-a}\sin a. \end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{r\cos ar}{r^4+4}\,dr = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{r\sin ar}{r^4+4}\,dr = \frac{\pi}{2}e^{-a}\sin a$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin a.}$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}
\int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz &= \int_0^\pi \frac{Re^{i\theta}e^{iaRe^{i\theta}}}{(Re^{i\theta})^4+4}(Rie^{i\theta}\,d\theta) \\
&= \int_0^\pi \frac{e^{iaR(\cos\theta+i\sin\theta)}}{R^4e^{i4\theta}+4}(R^2ie^{i2\theta}\,d\theta) \\
&= \int_0^\pi \frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{R^4e^{i4\theta}+4}(R^2ie^{i2\theta}\,d\theta)
\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
\left|\int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz\right| &= \left|\int_0^\pi \frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{R^4e^{i4\theta}+4}(R^2ie^{i2\theta}\,d\theta)\right| \\
&\leq \int_0^\pi \left|\frac{e^{iaR\cos\theta}e^{-aR\sin\theta}}{R^4e^{i4\theta}+4}(R^2ie^{i2\theta})\right| d\theta \\
&= \int_0^\pi \frac{\left|e^{iaR\cos\theta}\right|\left|e^{-aR\sin\theta}\right|}{|R^4e^{i4\theta}+4|}\left|R^2ie^{i2\theta}\right| d\theta \\
&= \int_0^\pi \frac{e^{-aR\sin\theta}}{|R^4e^{i4\theta}+4|}R^2\,d\theta \\
&\leq \int_0^\pi \frac{e^{-aR\sin\theta}}{|R^4e^{i4\theta}|-|4|}R^2\,d\theta \\
&= \int_0^\pi \frac{e^{-aR\sin\theta}}{R^4-4}R^2\,d\theta \\
&= \int_0^\pi \frac{e^{-aR\sin\theta}}{1-\frac{4}{R^4}}\frac{d\theta}{R^2}
\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz\right| \leq \lim_{R\to\infty}\int_0^\pi \frac{e^{-aR\sin\theta}}{1-\frac{4}{R^4}}\frac{d\theta}{R^2}$$

Because the limits of integration do not depend on R, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz\right| \leq \int_0^\pi \lim_{R\to\infty}\frac{e^{-aR\sin\theta}}{1-\frac{4}{R^4}}\frac{d\theta}{R^2}$$

Since θ lies between 0 and π, the sine of θ is positive. a is also positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz\right| \leq 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz\right| = 0 \quad \to \quad \lim_{R\to\infty}\int_{C_R} \frac{ze^{iaz}}{z^4+4}\,dz = 0.$$