

Exercise 8

Use residues to evaluate the improper integrals in Exercises 1 through 8.

$$\int_0^{\infty} \frac{x^3 \sin x \, dx}{(x^2 + 1)(x^2 + 9)}.$$

Solution

The integrand is an even function of x , so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^{\infty} \frac{x^3 \sin x \, dx}{(x^2 + 1)(x^2 + 9)} = \int_{-\infty}^{\infty} \frac{x^3 \sin x \, dx}{2(x^2 + 1)(x^2 + 9)}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)},$$

and the contour in Fig. 93. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} 2(z^2 + 1)(z^2 + 9) &= 0 \\ z^2 + 1 = 0 \quad \text{or} \quad z^2 + 9 = 0 \\ z = \pm i \quad \text{or} \quad z = \pm 3i \end{aligned}$$

The singular points of interest to us are the ones that lie within the closed contour, $z = i$ and $z = 3i$.

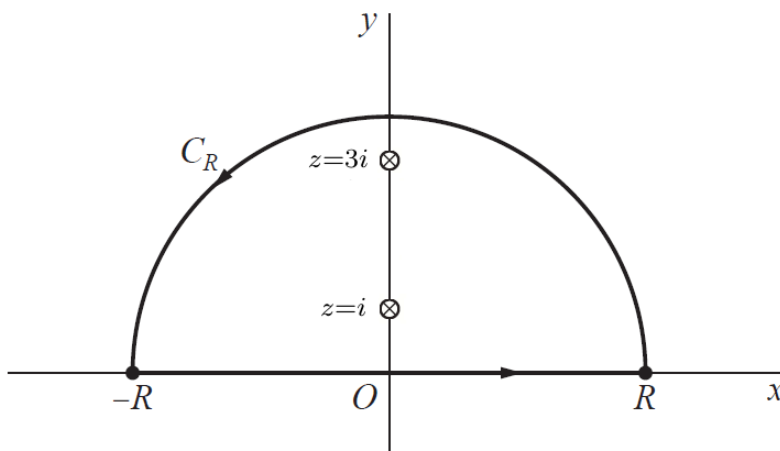


Figure 1: This is Fig. 93 with the singularities at $z = i$ and $z = 3i$ marked.

According to Cauchy's residue theorem, the integral of $z^3 e^{iz}/[2(z^2 + 1)(z^2 + 9)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} \, dz = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} + \operatorname{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} \right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\begin{aligned} \int_L \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} dz + \int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} dz \\ = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} + \operatorname{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} \right] \end{aligned}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: \quad z = r, \quad r = -R \rightarrow r = R \\ C_R: \quad z = Re^{i\theta}, \quad \theta = 0 \rightarrow \theta = \pi \end{aligned}$$

As a result,

$$\begin{aligned} \int_{-R}^R \frac{r^3 e^{ir}}{2(r^2+1)(r^2+9)} dr + \int_{C_R} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} dz \\ = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} + \operatorname{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} \right]. \end{aligned}$$

Take the limit now as $R \rightarrow \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{r^3 e^{ir}}{2(r^2+1)(r^2+9)} dr = 2\pi i \left[\operatorname{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} + \operatorname{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} \right]$$

The denominator can be written as $2(z^2+1)(z^2+9) = 2(z+i)(z-i)(z+3i)(z-3i)$. From this we see that the multiplicities of the $z-i$ and $z-3i$ factors are both 1. The residues at $z=i$ and $z=3i$ can then be calculated by

$$\begin{aligned} \operatorname{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} &= \phi_1(i) \\ \operatorname{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} &= \phi_2(3i), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z-i$ and $z-3i$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{z^3 e^{iz}}{2(z+i)(z+3i)(z-3i)} \Rightarrow \phi_1(i) = -\frac{1}{32} e^{-1} \\ \phi_2(z) &= \frac{z^3 e^{iz}}{2(z+i)(z-i)(z+3i)} \Rightarrow \phi_2(3i) = \frac{9}{32} e^{-3} \end{aligned}$$

So then

$$\begin{aligned} \operatorname{Res}_{z=i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} &= -\frac{1}{32} e^{-1} \\ \operatorname{Res}_{z=3i} \frac{z^3 e^{iz}}{2(z^2+1)(z^2+9)} &= \frac{9}{32} e^{-3} \end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{r^3 e^{ir}}{2(r^2+1)(r^2+9)} dr &= 2\pi i \left(-\frac{1}{32}e^{-1} + \frac{9}{32}e^{-3} \right) \\ \int_{-\infty}^{\infty} \frac{r^3 \cos r + ir^3 \sin r}{2(r^2+1)(r^2+9)} dr &= \frac{\pi i}{16} \left(\frac{9}{e^3} - \frac{1}{e} \right) \\ \int_{-\infty}^{\infty} \frac{r^3 \cos r}{2(r^2+1)(r^2+9)} dr + i \int_{-\infty}^{\infty} \frac{r^3 \sin r}{2(r^2+1)(r^2+9)} dr &= \frac{i\pi}{16} \left(\frac{9-e^2}{e^3} \right).\end{aligned}$$

Match the real and imaginary parts of both sides.

$$\int_{-\infty}^{\infty} \frac{r^3 \cos r}{2(r^2+1)(r^2+9)} dr = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{r^3 \sin r}{2(r^2+1)(r^2+9)} dr = \frac{\pi}{16} \left(\frac{9-e^2}{e^3} \right)$$

Therefore, changing the dummy integration variable to x ,

$$\boxed{\int_0^{\infty} \frac{x^3 \sin x dx}{(x^2+1)(x^2+9)} = \frac{\pi}{16} \left(\frac{9-e^2}{e^3} \right)}.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \rightarrow \infty$. The parameterization of the semicircular arc in Fig. 93 is $z = Re^{i\theta}$, where θ goes from 0 to π .

$$\begin{aligned} \int_{C_R} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} dz &= \int_0^\pi \frac{(Re^{i\theta})^3 e^{iRe^{i\theta}}}{2[(Re^{i\theta})^2 + 1][(Re^{i\theta})^2 + 9]} (Rie^{i\theta} d\theta) \\ &= \int_0^\pi \frac{e^{iR(\cos\theta + i\sin\theta)}}{2(R^2 e^{i2\theta} + 1)(R^2 e^{i2\theta} + 9)} (R^4 i e^{i4\theta} d\theta) \\ &= \int_0^\pi \frac{e^{iR\cos\theta} e^{-R\sin\theta}}{2(R^2 e^{i2\theta} + 1)(R^2 e^{i2\theta} + 9)} (R^4 i e^{i4\theta} d\theta) \end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned} \left| \int_{C_R} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} dz \right| &= \left| \int_0^\pi \frac{e^{iR\cos\theta} e^{-R\sin\theta}}{2(R^2 e^{i2\theta} + 1)(R^2 e^{i2\theta} + 9)} (R^4 i e^{i4\theta} d\theta) \right| \\ &\leq \int_0^\pi \left| \frac{e^{iR\cos\theta} e^{-R\sin\theta}}{2(R^2 e^{i2\theta} + 1)(R^2 e^{i2\theta} + 9)} (R^4 i e^{i4\theta}) \right| d\theta \\ &= \int_0^\pi \frac{|e^{iR\cos\theta}| |e^{-R\sin\theta}|}{2 |R^2 e^{i2\theta} + 1| |R^2 e^{i2\theta} + 9|} |R^4 i e^{i4\theta}| d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{2 |R^2 e^{i2\theta} + 1| |R^2 e^{i2\theta} + 9|} R^4 d\theta \\ &\leq \int_0^\pi \frac{e^{-R\sin\theta}}{(|R^2 e^{i2\theta}| - |1|)(|R^2 e^{i2\theta}| - |9|)} \frac{R^4}{2} d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{(R^2 - 1)(R^2 - 9)} \frac{R^4}{2} d\theta \\ &= \int_0^\pi \frac{e^{-R\sin\theta}}{\left(1 - \frac{1}{R^2}\right) \left(1 - \frac{9}{R^2}\right)} \frac{d\theta}{2} \end{aligned}$$

Now take the limit of both sides as $R \rightarrow \infty$.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} dz \right| \leq \lim_{R \rightarrow \infty} \int_0^\pi \frac{e^{-R\sin\theta}}{\left(1 - \frac{1}{R^2}\right) \left(1 - \frac{9}{R^2}\right)} \frac{d\theta}{2}$$

Because the limits of integration do not depend on R , the limit may be brought inside the integral.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} dz \right| \leq \int_0^\pi \lim_{R \rightarrow \infty} \frac{e^{-R\sin\theta}}{\left(1 - \frac{1}{R^2}\right) \left(1 - \frac{9}{R^2}\right)} \frac{d\theta}{2}$$

Since θ lies between 0 and π , the sine of θ is positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} dz \right| \leq 0$$

The magnitude of a number cannot be negative, and the only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} dz \right| = 0 \quad \rightarrow \quad \lim_{R \rightarrow \infty} \int_{C_R} \frac{z^3 e^{iz}}{2(z^2 + 1)(z^2 + 9)} dz = 0.$$