Exercise 1

Use residues to evaluate the definite integrals in Exercises 1 through 7.

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}.$$

$$Ans.\ \frac{2\pi}{3}.$$

Solution

Because the integral goes from 0 to 2π, it can be thought of as one over the unit circle in the complex plane.

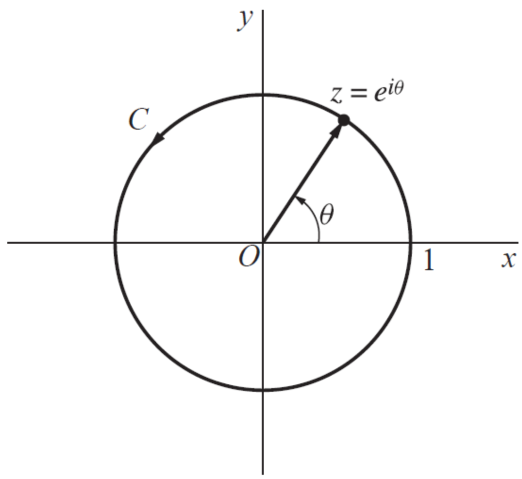

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

This circle is parameterized in terms of θ by $z = e^{i\theta} = \cos\theta + i\sin\theta$. Solve for $\sin\theta$ and $d\theta$ in terms of z and dz, respectively.

$$\begin{cases} z = e^{i\theta} = \cos\theta + i\sin\theta \\ z^{-1} = e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \quad \rightarrow \quad z - z^{-1} = 2i\sin\theta \quad \rightarrow \quad \sin\theta = \frac{z - z^{-1}}{2i}$$

$$z = e^{i\theta} \quad \rightarrow \quad dz = ie^{i\theta}\,d\theta = iz\,d\theta \quad \rightarrow \quad d\theta = \frac{dz}{iz}$$

With this change of variables the integral in $d\theta$ will become a positively oriented closed loop integral over the circle's boundary C.

$$\begin{aligned} \int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} &= \oint_C \frac{1}{5 + 4\left(\frac{z - z^{-1}}{2i}\right)} \frac{dz}{iz} \\ &= \oint_C \frac{dz}{5iz + 2z\left(z - z^{-1}\right)} \end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} &= \oint_C \frac{dz}{2z^2+5iz-2} \\ &= \oint_C \frac{dz}{(z+2i)(2z+i)} \\ &= \oint_C \frac{dz}{2(z+2i)\left(z+\frac{i}{2}\right)}\end{aligned}$$

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside C. Because there is only one singular point inside the unit circle, namely $z = -i/2$, there is only one residue to calculate.

$$\oint_C \frac{dz}{2(z+2i)\left(z+\frac{i}{2}\right)} = 2\pi i \operatorname*{Res}_{z=-\frac{i}{2}} \frac{1}{2(z+2i)\left(z+\frac{i}{2}\right)}$$

The multiplicity of the factor $z + i/2$ is 1, so the residue is calculated by

$$\operatorname*{Res}_{z=-\frac{i}{2}} \frac{1}{2(z+2i)\left(z+\frac{i}{2}\right)} = \phi\left(-\frac{i}{2}\right),$$

where $\phi(z)$ is the same function as the integrand without the factor $z + i/2$.

$$\phi(z) = \frac{1}{2(z+2i)}$$

So then

$$\operatorname*{Res}_{z=-\frac{i}{2}} \frac{1}{2(z+2i)\left(z+\frac{i}{2}\right)} = \frac{1}{2(-\frac{i}{2}+2i)} = \frac{1}{3i}$$

and

$$\oint_C \frac{dz}{2(z+2i)\left(z+\frac{i}{2}\right)} = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}.$$

Therefore,

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}.$$