Exercise 2

Use residues to evaluate the definite integrals in Exercises 1 through 7.

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}.$$
$$Ans.\ \sqrt{2}\pi.$$

Solution

Start off by making the substitution,

$$\begin{aligned} \alpha &= \theta + \pi \quad \rightarrow \quad \theta = \alpha - \pi \\ d\alpha &= d\theta, \end{aligned}$$

so that the integral goes from 0 to 2π.

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \int_{-\pi+\pi}^{\pi+\pi} \frac{d\alpha}{1+\sin^2(\alpha-\pi)} = \int_0^{2\pi} \frac{d\alpha}{1+\sin^2\alpha}$$

The integral can now be thought of as one over the unit circle in the complex plane.

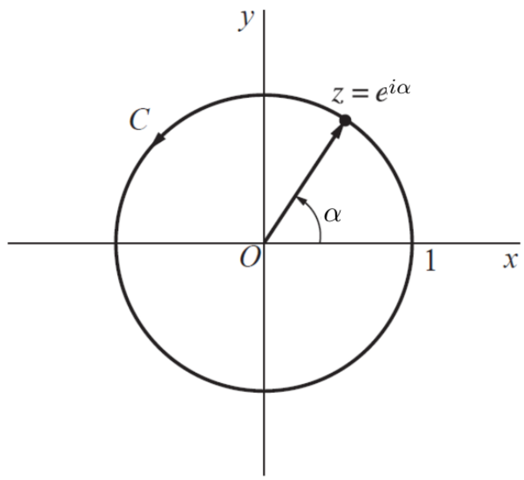

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

This circle is parameterized in terms of α by $z = e^{i\alpha} = \cos\alpha + i\sin\alpha$. Solve for $\sin\alpha$ and $d\alpha$ in terms of z and dz, respectively.

$$\begin{cases} z = e^{i\alpha} = \cos\alpha + i\sin\alpha \\ z^{-1} = e^{-i\alpha} = \cos\alpha - i\sin\alpha \end{cases} \quad \rightarrow \quad z - z^{-1} = 2i\sin\alpha \quad \rightarrow \quad \sin\alpha = \frac{z-z^{-1}}{2i}$$

$$z = e^{i\alpha} \quad \rightarrow \quad dz = ie^{i\alpha}\,d\alpha = iz\,d\alpha \quad \rightarrow \quad d\alpha = \frac{dz}{iz}$$

With this change of variables the integral in $d\alpha$ will become a positively oriented closed loop integral over the circle's boundary C.

$$\begin{aligned}\int_0^{2\pi} \frac{d\alpha}{1+\sin^2\alpha} &= \oint_C \frac{1}{1+\left(\frac{z-z^{-1}}{2i}\right)^2} \frac{dz}{iz} \\ &= \oint_C \frac{1}{\frac{3}{2}-\frac{1}{4z^2}-\frac{z^2}{4}} \frac{4iz\,dz}{4iz\cdot iz} \\ &= \oint_C \frac{4iz\,dz}{z^4-6z^2+1}\end{aligned}$$

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside C. Determine the four singular points of the integrand by solving for the roots of the denominator.

$$z^4-6z^2+1=0$$

$$z^2 = \frac{6\pm\sqrt{36-4}}{2} = 3\pm 2\sqrt{2} \quad \rightarrow \quad \begin{cases} z_1 = -\sqrt{3+2\sqrt{2}} = -1-\sqrt{2} \approx -2.414 \\ z_2 = -\sqrt{3-2\sqrt{2}} = 1-\sqrt{2} \approx -0.414 \\ z_3 = \sqrt{3-2\sqrt{2}} = -1+\sqrt{2} \approx 0.414 \\ z_4 = \sqrt{3+2\sqrt{2}} = 1+\sqrt{2} \approx 2.414 \end{cases}$$

Because there are only two singular points inside the unit circle, namely $z=z_2$ and $z=z_3$, there are only two residues to calculate.

$$\oint_C \frac{4iz\,dz}{z^4-6z^2+1} = 2\pi i\left(\operatorname*{Res}_{z=z_2} \frac{4iz}{z^4-6z^2+1} + \operatorname*{Res}_{z=z_3} \frac{4iz}{z^4-6z^2+1}\right)$$

The denominator can be factored as $z^4-6z^2+1=(z-z_1)(z-z_2)(z-z_3)(z-z_4)$. From this we see that the multiplicities of $z-z_2$ and $z-z_3$ are both 1, so the residues are calculated by

$$\begin{aligned}\operatorname*{Res}_{z=z_2} \frac{4iz}{z^4-6z^2+1} &= \phi_2(z_2) \\ \operatorname*{Res}_{z=z_3} \frac{4iz}{z^4-6z^2+1} &= \phi_3(z_3),\end{aligned}$$

where $\phi_2(z)$ and $\phi_3(z)$ are the same function as the integrand without the factors $z-z_2$ and $z-z_3$, respectively.

$$\begin{aligned}\phi_2(z) &= \frac{4iz}{(z-z_1)(z-z_3)(z-z_4)} \\ \phi_3(z) &= \frac{4iz}{(z-z_1)(z-z_2)(z-z_4)}\end{aligned}$$

So then

$$\begin{aligned}\operatorname*{Res}_{z=z_2} \frac{4iz}{z^4-6z^2+1} &= \frac{4iz_2}{(z_2-z_1)(z_2-z_3)(z_2-z_4)} = \frac{4i(1-\sqrt{2})}{2(2-2\sqrt{2})(-2\sqrt{2})} = -\frac{i\sqrt{2}}{4} \\ \operatorname*{Res}_{z=z_3} \frac{4iz}{z^4-6z^2+1} &= \frac{4iz_3}{(z_3-z_1)(z_3-z_2)(z_3-z_4)} = \frac{4i(-1+\sqrt{2})}{(2\sqrt{2})(-2+2\sqrt{2})(-2)} = -\frac{i\sqrt{2}}{4}\end{aligned}$$

and

$$\oint_C \frac{4iz\,dz}{z^4 - 6z^2 + 1} = 2\pi i \left(-\frac{i\sqrt{2}}{4} - \frac{i\sqrt{2}}{4} \right) = \sqrt{2}\pi$$

and

$$\int_0^{2\pi} \frac{d\alpha}{1 + \sin^2 \alpha} = \sqrt{2}\pi.$$

Therefore,

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} = \sqrt{2}\pi.$$