Exercise 7

Use residues to evaluate the definite integrals in Exercises 1 through 7.

$$\int_0^{\pi} \sin^{2n}\theta\, d\theta \quad (n = 1, 2, \ldots).$$

$$Ans.\ \frac{(2n)!}{2^{2n}(n!)^2}\pi.$$

Solution

Notice that the integrand is an even function of θ, so the lower limit of integration can be extended to $-\pi$ as long as the integral is divided by 2.

$$\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta\, d\theta$$

Now make the substitution,

$$\alpha = \theta + \pi \quad \rightarrow \quad \theta = \alpha - \pi$$
$$d\alpha = d\theta,$$

so that the integral goes from 0 to 2π.

$$\frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta\, d\theta = \frac{1}{2}\int_{-\pi+\pi}^{\pi+\pi} \sin^{2n}(\alpha - \pi)\, d\alpha = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\alpha\, d\alpha$$

The integral can now be thought of as one over the unit circle in the complex plane.

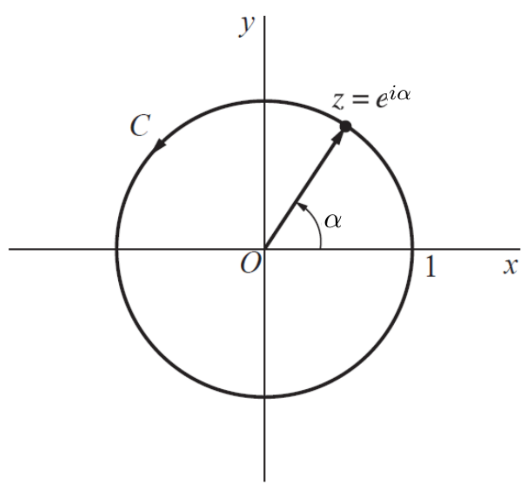

Figure 1: This figure illustrates the unit circle in the complex plane, where $z = x + iy$.

This circle is parameterized in terms of α by $z = e^{i\alpha} = \cos\alpha + i\sin\alpha$. Solve for $\sin\alpha$ and $d\alpha$ in terms of z and dz, respectively.

$$\begin{cases} z = e^{i\alpha} = \cos\alpha + i\sin\alpha \\ z^{-1} = e^{-i\alpha} = \cos\alpha - i\sin\alpha \end{cases} \quad \rightarrow \quad z - z^{-1} = 2i\sin\alpha \quad \rightarrow \quad \sin\alpha = \frac{z - z^{-1}}{2i}$$

$$z = e^{i\alpha} \quad \rightarrow \quad dz = ie^{i\alpha}\, d\alpha = iz\, d\alpha \quad \rightarrow \quad d\alpha = \frac{dz}{iz}$$

With this change of variables the integral in $d\alpha$ will become a positively oriented closed loop integral over the circle's boundary C.

$$\begin{aligned}\frac{1}{2}\int_0^{2\pi}\sin^{2n}\alpha\,d\alpha &= \oint_C \frac{1}{2}\left(\frac{z-z^{-1}}{2i}\right)^{2n}\frac{dz}{iz}\\ &= \oint_C \frac{1}{(2i)^{2n+1}}\left(\frac{z^2-1}{z}\right)^{2n}\frac{dz}{z}\\ &= \oint_C \frac{1}{(2i)^{2n+1}}\frac{(z^2-1)^{2n}}{z^{2n+1}}\,dz\end{aligned}$$

According to the Cauchy residue theorem, such an integral in the complex plane is equal to $2\pi i$ times the sum of the residues inside C. Because there is only one singular point inside the unit circle, namely $z=0$, there is only one residue to calculate.

$$\oint_C \frac{1}{(2i)^{2n+1}}\frac{(z^2-1)^{2n}}{z^{2n+1}}\,dz = 2\pi i \operatorname*{Res}_{z=0}\frac{1}{(2i)^{2n+1}}\frac{(z^2-1)^{2n}}{z^{2n+1}}$$

The multiplicity of the factor z is $2n+1$, so the residue is calculated by

$$\operatorname*{Res}_{z=0}\frac{1}{(2i)^{2n+1}}\frac{(z^2-1)^{2n}}{z^{2n+1}} = \frac{\phi^{(2n+1-1)}(0)}{(2n+1-1)!} = \frac{\phi^{(2n)}(0)}{(2n)!},$$

where $\phi(z)$ is the same function as the integrand without z^{2n+1}.

$$\phi(z) = \frac{1}{(2i)^{2n+1}}(z^2-1)^{2n}$$

In order to figure out what $\phi^{(2n)}(z)$ is, take derivatives of it until a pattern becomes apparent.

$$\begin{aligned}
n=1: &\quad \phi^{(2)}(z) = \frac{1}{(2i)^{2n+1}}2*2n*\left[-1+z^2(4n-1)\right]\\
n=2: &\quad \phi^{(4)}(z) = \frac{1}{(2i)^{2n+1}}4*2n(2n-1)*\left\{3+z^2(4n-3)\left[-6+(4n-1)z^2\right]\right\}\\
n=3: &\quad \phi^{(6)}(z) = \frac{1}{(2i)^{2n+1}}8*2n(2n-1)(2n-2)*\left\{-15+z^2\left[\ldots\right]\right\}\\
n=4: &\quad \phi^{(8)}(z) = \frac{1}{(2i)^{2n+1}}16*2n(2n-1)(2n-2)(2n-3)*\left\{105+z^2\left[\ldots\right]\right\}\\
&\quad\vdots \qquad\qquad\qquad\qquad \vdots
\end{aligned}$$

$$\begin{aligned}\phi^{(2n)}(0) &= \frac{1}{(2i)^{2n+1}}2^n*\frac{(2n)!}{n!}*\left[(-1)^n(2n-1)!!\right]\\ &= \frac{1}{(2i)^{2n+1}}2^n*\frac{(2n)!}{n!}*\left[(i^2)^n\frac{(2n)!}{2^n n!}\right]\\ &= \frac{1}{2i}\frac{1}{2^{2n}i^{2n}}i^{2n}\frac{[(2n)!]^2}{(n!)^2}\\ &= \frac{1}{2i}\frac{[(2n)!]^2}{2^{2n}(n!)^2}\end{aligned}$$

So then

$$\operatorname*{Res}_{z=0}\frac{1}{(2i)^{2n+1}}\frac{(z^2-1)^{2n}}{z^{2n+1}} = \frac{\phi^{(2n)}(0)}{(2n)!} = \frac{1}{2i}\frac{(2n)!}{2^{2n}(n!)^2}$$

and

$$\oint_C \frac{1}{(2i)^{2n+1}} \frac{(z^2-1)^{2n}}{z^{2n+1}}\, dz = 2\pi i \left[\frac{1}{2i} \frac{(2n)!}{2^{2n}(n!)^2}\right] = \frac{(2n)!}{2^{2n}(n!)^2}\pi.$$

Therefore,

$$\int_0^\pi \sin^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi \quad (n = 1, 2, \ldots).$$