Exercise 13

Let the point $s_0 = \alpha + i\beta$ $(\beta \neq 0)$ be a pole of order m of a function $F(s)$, which has a Laurent series representation

$$F(s) = \sum_{n=0}^{\infty} a_n(s - s_0)^n + \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{b_m}{(s - s_0)^m} \quad (b_m \neq 0)$$

in the punctured disk $0 < |s - s_0| < R_2$. Also, assume that $\overline{F(s)} = F(\bar{s})$ at points s where $F(s)$ is analytic.

(a) With the aid of the result in Exercise 6, Sec. 56, point out how it follows that

$$F(\bar{s}) = \sum_{n=0}^{\infty} \overline{a_n}(\bar{s} - \overline{s_0})^n + \frac{\overline{b_1}}{\bar{s} - \overline{s_0}} + \frac{\overline{b_2}}{(\bar{s} - \overline{s_0})^2} + \cdots + \frac{\overline{b_m}}{(\bar{s} - \overline{s_0})^m} \quad (\overline{b_m} \neq 0)$$

when $0 < |\bar{s} - \overline{s_0}| < R_2$. Then replace $\bar{s}$ by s here to obtain a Laurent series representation for $F(s)$ in the punctured disk $0 < |s - \overline{s_0}| < R_2$, and conclude that $\overline{s_0}$ is a pole of order m of $F(s)$.

(b) Use results in Exercise 12 and part (a) to show that

$$\operatorname*{Res}_{s=s_0}\,[e^{st}F(s)] + \operatorname*{Res}_{s=\overline{s_0}}\,[e^{st}F(s)] = 2e^{\alpha t}\,\text{Re}\left\{e^{i\beta t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_m}{(m-1)!}t^{m-1}\right]\right\}$$

when t is real, as stated just before Example 1 in Sec. 89.

Solution

Part (a)

The result of Exercise 6 in Sec. 56 says that

$$\text{if} \quad \sum_{n=1}^{\infty} z_n = S, \quad \text{then} \quad \sum_{n=1}^{\infty} \overline{z_n} = \overline{S}.$$

We have

$$F(s) = \sum_{n=0}^{\infty} a_n(s - s_0)^n + \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{b_m}{(s - s_0)^m}$$

when $0 < |s - s_0| < R_2$. Take the complex conjugate of both sides.

$$\overline{F(s)} = \overline{\sum_{n=0}^{\infty} a_n(s - s_0)^n + \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{b_m}{(s - s_0)^m}}$$

It is assumed that $\overline{F(s)} = F(\bar{s})$.

$$\begin{aligned} F(\bar{s}) &= \overline{\sum_{n=0}^{\infty} a_n(s - s_0)^n + \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{b_m}{(s - s_0)^m}} \\ &= \overline{\sum_{n=0}^{\infty} a_n(s - s_0)^n} + \overline{\frac{b_1}{s - s_0}} + \overline{\frac{b_2}{(s - s_0)^2}} + \cdots + \overline{\frac{b_m}{(s - s_0)^m}} \end{aligned}$$

Apply the result from Exercise 6 in Sec. 56 here.

$$\begin{aligned} F(\bar{s}) &= \sum_{n=0}^{\infty} \overline{a_n(s-s_0)^n} + \overline{\frac{b_1}{s-s_0}} + \overline{\frac{b_2}{(s-s_0)^2}} + \cdots + \overline{\frac{b_m}{(s-s_0)^m}} \\ &= \sum_{n=0}^{\infty} \overline{a_n}\,\overline{(s-s_0)^n} + \frac{\overline{b_1}}{\overline{s-s_0}} + \frac{\overline{b_2}}{\overline{(s-s_0)^2}} + \cdots + \frac{\overline{b_m}}{\overline{(s-s_0)^m}} \\ &= \sum_{n=0}^{\infty} \overline{a_n}(\overline{s-s_0})^n + \frac{\overline{b_1}}{\bar{s}-\overline{s_0}} + \frac{\overline{b_2}}{(\overline{s-s_0})^2} + \cdots + \frac{\overline{b_m}}{(\overline{s-s_0})^m} \end{aligned}$$

Therefore,

$$F(\bar{s}) = \sum_{n=0}^{\infty} \overline{a_n}(\bar{s}-\overline{s_0})^n + \frac{\overline{b_1}}{\bar{s}-\overline{s_0}} + \frac{\overline{b_2}}{(\bar{s}-\overline{s_0})^2} + \cdots + \frac{\overline{b_m}}{(\bar{s}-\overline{s_0})^m}$$

when $0 < |\bar{s}-\overline{s_0}| < R_2$. Now replace $\bar{s}$ with s to obtain a Laurent series representation for $F(s)$ in the punctured disk $0 < |s-\overline{s_0}| < R_2$.

$$F(s) = \sum_{n=0}^{\infty} \overline{a_n}(s-\overline{s_0})^n + \frac{\overline{b_1}}{s-\overline{s_0}} + \frac{\overline{b_2}}{(s-\overline{s_0})^2} + \cdots + \frac{\overline{b_m}}{(s-\overline{s_0})^m}$$

Therefore, $\overline{s_0}$ is a pole of order m.

Part (b)

From Exercise 12 we know that if $F(s)$ has a pole at $s = s_0$ of order m, then

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)] = e^{s_0 t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right].$$

From part (a) we know that $F(s)$ also has a pole at $s = \overline{s_0}$ of order m.

$$\operatorname*{Res}_{s=\overline{s_0}}[e^{st}F(s)] = e^{\overline{s_0}t}\left[\overline{b_1} + \frac{\overline{b_2}}{1!}t + \cdots + \frac{\overline{b_{m-1}}}{(m-2)!}t^{m-2} + \frac{\overline{b_m}}{(m-1)!}t^{m-1}\right]$$

m and t are real, so the complex conjugate extends over the terms that include them.

$$\begin{aligned} &= e^{\overline{s_0 t}}\left[\overline{b_1} + \overline{\frac{b_2}{1!}t} + \cdots + \overline{\frac{b_{m-1}}{(m-2)!}t^{m-2}} + \overline{\frac{b_m}{(m-1)!}t^{m-1}}\right] \\ &= \overline{e^{s_0 t}}\,\overline{\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right]} \\ &= \overline{e^{s_0 t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right]} \end{aligned}$$

Thus, the sum of the two residues is

$$\begin{aligned} \operatorname*{Res}_{s=s_0}[e^{st}F(s)] + \operatorname*{Res}_{s=\overline{s_0}}[e^{st}F(s)] &= e^{s_0 t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right] \\ &\quad + \overline{e^{s_0 t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right]}. \end{aligned}$$

Make use of the fact that $z + \bar{z} = 2\operatorname{Re} z$.

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)] + \operatorname*{Res}_{s=\overline{s_0}}[e^{st}F(s)] = 2\operatorname{Re}\left\{e^{s_0 t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right]\right\}$$

Substitute $s_0 = \alpha + i\beta$ on the right side.

$$= 2\operatorname{Re}\left\{e^{(\alpha+i\beta)t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right]\right\}$$

$$= 2\operatorname{Re}\left\{e^{\alpha t}e^{i\beta t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right]\right\}$$

Since α and t are real, $e^{\alpha t}$ can be pulled in front of Re. Therefore,

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)] + \operatorname*{Res}_{s=\overline{s_0}}[e^{st}F(s)] = 2e^{\alpha t}\operatorname{Re}\left\{e^{i\beta t}\left[b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_m}{(m-1)!}t^{m-1}\right]\right\}.$$