Exercise 13

Let the point $s_0 = \alpha + i\beta$ $(\beta \neq 0)$ be a pole of order m of a function $F(s)$, which has a Laurent series representation

$$F(s) = \sum_{n=0}^{\infty} a_n (s - s_0)^n + \frac{b_1}{s - s_0} + \frac{b_2}{(s - s_0)^2} + \cdots + \frac{b_m}{(s - s_0)^m} \quad (b_m \neq 0)$$

in the punctured disk $0 < |s - s_0| < R_2$. Also, assume that $\overline{F(s)} = F(\bar{s})$ at points s where $F(s)$ is analytic.

(a) With the aid of the result in Exercise 6, Sec. 56, point out how it follows that

$$F(\bar{s}) = \sum_{n=0}^{\infty} \overline{a_n} (\bar{s} - \overline{s_0})^n + \frac{\overline{b_1}}{\bar{s} - \overline{s_0}} + \frac{\overline{b_2}}{(\bar{s} - \overline{s_0})^2} + \cdots + \frac{\overline{b_m}}{(\bar{s} - \overline{s_0})^m} \quad (\overline{b_m} \neq 0)$$

when $0 < |\bar{s} - \overline{s_0}| < R_2$. Then replace $\bar{s}$ by s here to obtain a Laurent series representation for $F(s)$ in the punctured disk $0 < |s - \overline{s_0}| < R_2$, and conclude that $\overline{s_0}$ is a pole of order m of $F(s)$.

(b) Use results in Exercise 12 and part (a) to show that

$$\operatorname*{Res}_{s=s_0} [e^{st} F(s)] + \operatorname*{Res}_{s=\overline{s_0}} [e^{st} F(s)] = 2e^{\alpha t} \operatorname{Re} \left\{ e^{i\beta t} \left[b_1 + \frac{b_2}{1!} t + \cdots + \frac{b_m}{(m-1)!} t^{m-1} \right] \right\}$$

when t is real, as stated just before Example 1 in Sec. 89.