Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

(a) $4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

Solution

$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 4$, $B = 5$, $C = 1$, $D = 1$, $E = 1$, $F = 0$, and $G = 2$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{8}\left(5 \pm \sqrt{25 - 16}\right) \\
\frac{dy}{dx} &= \frac{1}{8}(5 \pm 3) \\
\frac{dy}{dx} &= 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{4}.
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 25 - 16 = 9$, is greater than 0, which means that the PDE is **hyperbolic**. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane.

$$y = x + C_1 \quad \text{or} \quad y = \frac{1}{4}x + C_2.$$

Solving for the constants of integration,

$$\begin{aligned}
C_1 &= y - x = \phi(x, y) \\
C_2 &= y - \frac{1}{4}x = \psi(x, y).
\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x, y) = y - x$ and $\eta = \psi(x, y) = y - \frac{1}{4}x$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -\frac{9}{4}$, $C^* = 0$, $D^* = 0$, $E^* = \frac{3}{4}$, $F^* = 0$, and $G^* = 2$. Thus, the PDE simplifies to

$$-\frac{9}{4}u_{\xi\eta} + \frac{3}{4}u_\eta = 2.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = \frac{1}{3}u_\eta - \frac{8}{9}.$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{3}(u_\alpha - u_\beta) - \frac{8}{9}.$$

This is the second canonical form of the hyperbolic PDE.