Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

(c) $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$

Solution

$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = y$, $B = x + y$, $C = x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{2y}\left(x + y \pm \sqrt{(x+y)^2 - 4xy}\right) \\
\frac{dy}{dx} &= \frac{1}{2y}\left(x + y \pm \sqrt{(x-y)^2}\right) \\
\frac{dy}{dx} &= \frac{1}{2y}\left(x + y \pm |x-y|\right).
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = (x-y)^2$, is greater than 0 for all x and y, which means that the PDE is **hyperbolic**. Therefore, the solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane.

$$\begin{aligned}
x > y \Rightarrow |x-y| = x - y: \quad & \frac{dy}{dx} = \frac{1}{2y}(2x) = \frac{x}{y}, \quad \frac{dy}{dx} = \frac{1}{2y}(2y) = 1 \\
x < y \Rightarrow |x-y| = y - x: \quad & \frac{dy}{dx} = \frac{1}{2y}(2y) = 1, \quad \frac{dy}{dx} = \frac{1}{2y}(2x) = \frac{x}{y}
\end{aligned}$$

The two characteristic equations are the same regardless of whether x is greater than y or not:

$$\frac{dy}{dx} = \frac{x}{y} \quad \text{or} \quad \frac{dy}{dx} = 1.$$

Integrating these equations, we find that

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1 \quad \text{or} \quad y = x + C_2.$$

Solving for the constants of integration (or any convenient multiple thereof),

$$\begin{aligned}
2C_1 &= y^2 - x^2 = \phi(x,y) \\
C_2 &= y - x = \psi(x,y).
\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x,y) = y^2 - x^2$ and $\eta = \psi(x,y) = y - x$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$
\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}
$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -2(x-y)^2 = -2\eta^2$, $C^* = 0$, $D^* = 2(x-y) = -2\eta$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-2\eta^2 u_{\xi\eta} - 2\eta u_\xi = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = -\frac{1}{\eta} u_\xi.$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. Solving for η gives $(\alpha - \beta)/2$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = -2\frac{u_\alpha + u_\beta}{\alpha - \beta}.$$

This is the second canonical form of the hyperbolic PDE.