Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

(d) $u_{xx} + yu_{yy} = 0$

Solution

$u_{xx} + yu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = y$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\pm\sqrt{-4y}\right)$$
$$\frac{dy}{dx} = \pm\sqrt{-y}.$$

Note that the discriminant, $B^2 - 4AC = -4y$, can be positive, zero, or negative, depending on whether $y < 0$, $y = 0$, or $y > 0$, respectively. That is,

$$\text{The PDE is } \begin{cases} \text{hyperbolic} & \text{if } y < 0. \\ \text{parabolic} & \text{if } y = 0. \\ \text{elliptic} & \text{if } y > 0. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $(y < 0)$

The ordinary differential equations yield one real family of characteristic curves in the xy-plane. Separating variables and integrating both sides of the characteristic equations, we find that

$$2\sqrt{-y} = \pm x + C_0.$$

Solving for y, the characteristic curves are $y(x) = -\frac{1}{4}(x \pm C_0)^2$. Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -x: \quad +C_0 = x + 2\sqrt{-y} = \phi(x, y)$$
$$\text{Working with } +x: \quad -C_0 = x - 2\sqrt{-y} = \psi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = x + 2\sqrt{-y}$ and $\eta = \psi(x, y) = x - 2\sqrt{-y}$, so that the PDE takes the simplest form. Solving these two equations for x and $2\sqrt{-y}$ gives
$x = (\xi + \eta)/2$ and $2\sqrt{-y} = (\xi - \eta)/2$. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 4$, $C^* = 0$, $D^* = \frac{1}{2\sqrt{-y}} = \frac{2}{\xi-\eta}$, $E^* = -\frac{1}{2\sqrt{-y}} = -\frac{2}{\xi-\eta}$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$4u_{\xi\eta} + \frac{2}{\xi-\eta}u_\xi - \frac{2}{\xi-\eta}u_\eta = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = -\frac{1}{2(\xi-\eta)}(u_\xi - u_\eta).$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = -\frac{1}{\beta}u_\beta.$$

This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(y = 0)$

Substituting $y = 0$ into the PDE reduces it immediately to the canonical form of a parabolic equation, $u_{xx} = 0$. The characteristic equation is given by

$$\frac{dy}{dx} = 0.$$

Solving this equation for y gives $y(x) = D$, where D is an arbitrary constant. The characteristic curves in the xy-plane are lines parallel to the x-axis.

Case III: The PDE is elliptic $(y > 0)$

Since the discriminant is negative for $y > 0$, the characteristic equations have no real solutions. This means that the family of characteristic curves lies in the complex plane:

$$\frac{dy}{dx} = \pm i\sqrt{y}.$$

Separating variables, integrating, using the fact that $1/i = -i$, and multiplying both sides by -1 gives

$$2i\sqrt{y} = \mp x + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned} \text{Working with } -x: &\quad +C_0 = x + 2i\sqrt{y} = \phi(x,y) \\ \text{Working with } +x: &\quad -C_0 = x - 2i\sqrt{y} = \psi(x,y). \end{aligned}$$

Because these functions are complex, however, the PDE will not be in the simplest form. Since ξ and η are complex conjugates of each other, we introduce the new real variables,

$$\begin{aligned} \alpha &= \frac{1}{2}(\xi + \eta) = x \\ \beta &= \frac{1}{2i}(\xi - \eta) = 2\sqrt{y}, \end{aligned}$$

which transform the PDE to the canonical form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_{\alpha} + E^{**}u_{\beta} + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned} A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\ B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\ C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\ D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\ E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\ F^{**} &= F \\ G^{**} &= G. \end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^{**} = 1$, $B^{**} = 0$, $C^{**} = 1$, $D^{**} = 0$, $E^{**} = -\frac{1}{2\sqrt{y}} = -\frac{1}{\beta}$, $F^{**} = 0$, and $G^{**} = 0$. The PDE becomes

$$u_{\alpha\alpha} + u_{\beta\beta} - \frac{1}{\beta}u_{\beta} = 0.$$

Solving for $u_{\alpha\alpha} + u_{\beta\beta}$ gives

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{\beta}u_{\beta}.$$

This is the canonical form of the elliptic PDE.