Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

(e) $yu_{xx} - 2u_{xy} + e^x u_{yy} + x^2 u_x - u = 0$

Solution

$yu_{xx} - 2u_{xy} + e^x u_{yy} + x^2 u_x - u = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = y$, $B = -2$, $C = e^x$, $D = x^2$, $E = 0$, $F = -1$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2y}\left(-2 \pm \sqrt{4 - 4ye^x}\right)$$
$$\frac{dy}{dx} = \frac{1}{y}\left(-1 \pm \sqrt{1 - ye^x}\right).$$

Note that the discriminant, $B^2 - 4AC = 4 - 4ye^x$, can be positive, zero, or negative, depending on whether $y < e^{-x}$, $y = e^{-x}$, or $y > e^{-x}$, respectively. That is,

$$\text{The PDE is } \begin{cases} \text{hyperbolic} & \text{if } y < e^{-x}. \\ \text{parabolic} & \text{if } y = e^{-x}. \\ \text{elliptic} & \text{if } y > e^{-x}. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic ($y < e^{-x}$)

The solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane. Unfortunately, the equations are difficult (if not impossible) to solve analytically, so the canonical form of the PDE cannot be determined. The characteristic curves can be visualized, however, by plotting the slope fields for each ODE; they are tangent to the slope fields at each point on the graph. See the figures on the following page. $y(x) = e^{-x}$ is plotted in red on each graph to show the boundary of the domain of hyperbolicity.

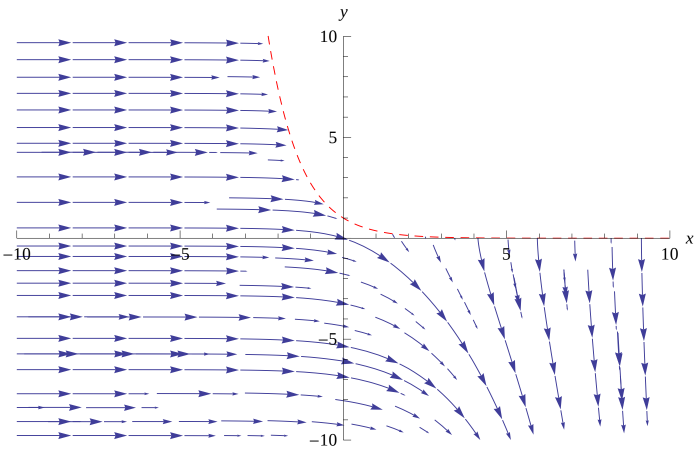

Figure 1: Slope field for $\frac{dy}{dx} = \frac{1}{y}\left(-1 + \sqrt{1 - ye^x}\right)$

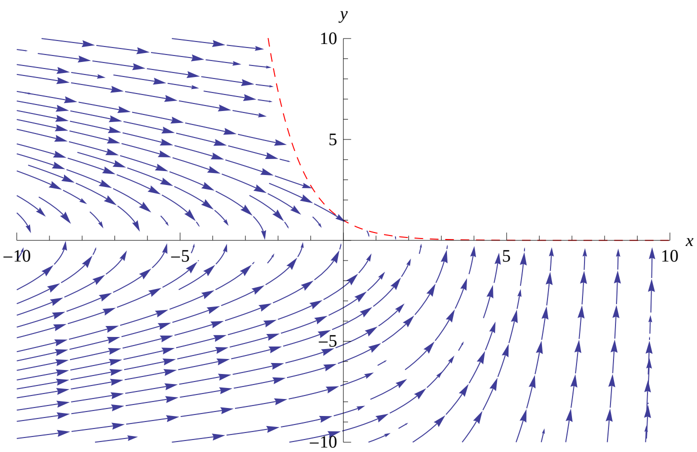

Figure 2: Slope field for $\frac{dy}{dx} = \frac{1}{y}\left(-1 - \sqrt{1 - ye^x}\right)$

Case II: The PDE is parabolic $(y = e^{-x})$

When $y = e^{-x}$, the characteristic equations reduce to

$$\frac{dy}{dx} = -\frac{1}{y} = -e^{x},$$

and this equation can be solved. Separating variables and integrating gives

$$y = -e^{x} + C_0.$$

Solving for the constant of integration,

$$C_0 = y + e^{x} = \phi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = y + e^{x}$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_{\xi} + E^* u_{\eta} + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 0$, $C^* = e^{x} = \xi - \eta$, $D^* = e^{x}(x^2 + e^{-x}) = x^2 e^{x} + 1 = [\ln(\xi - \eta)]^2(\xi - \eta) + 1$, $E^* = 0$, $F^* = -1$, and $G^* = 0$. Thus, the PDE simplifies to

$$(\xi - \eta)u_{\eta\eta} + \left\{(\xi - \eta)[\ln(\xi - \eta)]^2 + 1\right\} u_{\xi} - u = 0.$$

Solving for $u_{\eta\eta}$ gives

$$u_{\eta\eta} = -\left\{[\ln(\xi - \eta)]^2 + \frac{1}{\xi - \eta}\right\} u_{\xi} + \frac{1}{\xi - \eta} u.$$

This is the canonical form of the parabolic PDE.

Case III: The PDE is elliptic $(y > e^{-x})$

When $y > e^{-x}$, the characteristic equations satisfy

$$\frac{dy}{dx} = \frac{1}{y}\left(-1 \pm i\sqrt{ye^{x} - 1}\right),$$

and the two distinct families of characteristic curves lie in the complex plane. If we could solve the equations, we could determine the canonical form of the PDE.