

Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

$$(f) \quad u_{xx} + xu_{yy} = 0$$

Solution

$$u_{xx} + xu_{yy} = 0$$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2A} \left(B \pm \sqrt{B^2 - 4AC} \right) \\ \frac{dy}{dx} &= \frac{1}{2} (\pm \sqrt{-4x}) \\ \frac{dy}{dx} &= \pm \sqrt{-x}. \end{aligned}$$

Note that the discriminant, $B^2 - 4AC = -4x$, can be positive, zero, or negative, depending on whether $x < 0$, $x = 0$, or $x > 0$, respectively. That is,

$$\text{The PDE is } \begin{cases} \text{hyperbolic} & \text{if } x < 0. \\ \text{parabolic} & \text{if } x = 0. \\ \text{elliptic} & \text{if } x > 0. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic ($x < 0$)

The solutions to these ordinary differential equations are two families of real and distinct characteristic curves in the xy -plane. Integrating the equations, we find that

$$y(x) = \pm \frac{2}{3}(-x)^{3/2} + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof)¹,

$$\text{Working with } -\frac{2}{3}(-x)^{3/2}: \quad C_0 = y + \frac{2}{3}(-x)^{3/2} = \phi(x, y)$$

$$\text{Working with } +\frac{2}{3}(-x)^{3/2}: \quad C_0 = y - \frac{2}{3}(-x)^{3/2} = \psi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = y + \frac{2}{3}(-x)^{3/2}$ and $\eta = \psi(x, y) = y - \frac{2}{3}(-x)^{3/2}$, so that the PDE takes the simplest form. Solving these two equations for $(-x)^{3/2}$ and y gives $(-x)^{3/2} = \frac{3}{4}(\xi - \eta)$ and $y = \frac{1}{2}(\xi + \eta)$. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_{\xi} + E^*u_{\eta} + F^*u = G^*,$$

¹The book chooses to use $\frac{3}{2}C_0$ in the back, but this does not change the canonical form of the PDE.

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned} A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\ D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\ E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\ F^* &= F \\ G^* &= G. \end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 4x$, $C^* = 0$, $D^* = \frac{1}{2\sqrt{-x}}$, $E^* = -\frac{1}{2\sqrt{-x}}$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$\begin{aligned} 4xu_{\xi\eta} + \frac{1}{2\sqrt{-x}}u_{\xi} - \frac{1}{2\sqrt{-x}}u_{\eta} &= 0 \\ u_{\xi\eta} &= -\frac{1}{4x} \left(\frac{1}{2\sqrt{-x}}u_{\xi} - \frac{1}{2\sqrt{-x}}u_{\eta} \right) \\ u_{\xi\eta} &= \frac{1}{8(-x)^{3/2}}(u_{\xi} - u_{\eta}) \\ u_{\xi\eta} &= \frac{1}{6(\xi - \eta)}(u_{\xi} - u_{\eta}). \end{aligned}$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_{\xi} = u_{\alpha} + u_{\beta}$, and $u_{\eta} = u_{\alpha} - u_{\beta}$. The PDE then becomes

$$\begin{aligned} u_{\alpha\alpha} - u_{\beta\beta} &= \frac{1}{6\beta}[(u_{\alpha} + u_{\beta}) - (u_{\alpha} - u_{\beta})] \\ u_{\alpha\alpha} - u_{\beta\beta} &= \frac{1}{3\beta}u_{\beta}. \end{aligned}$$

This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic ($x = 0$)

Substituting $x = 0$ into the PDE reduces it immediately to the canonical form of a parabolic equation, $u_{xx} = 0$. The characteristic equation is given by

$$\frac{dy}{dx} = 0.$$

Solving this equation for y gives $y(x) = D$, where D is an arbitrary constant. The characteristic curves in the xy -plane are lines parallel to the x -axis.

Case III: The PDE is elliptic ($x > 0$)

The characteristic equations have no real solutions for $x > 0$. This means that the two distinct families of characteristic curves lie in the complex plane. Integrating the characteristic equations, we find that

$$\frac{dy}{dx} = \pm i\sqrt{x}$$

$$y(x) = \pm \frac{2i}{3}x^{3/2} + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\frac{2i}{3}x^{3/2}: \quad + C_0 = y + \frac{2i}{3}x^{3/2} = \phi(x, y)$$

$$\text{Working with } +\frac{2i}{3}x^{3/2}: \quad + C_0 = y - \frac{2i}{3}x^{3/2} = \psi(x, y).$$

The typical variables, $\xi = \phi(x, y) = y + \frac{2i}{3}x^{3/2}$ and $\eta = \psi(x, y) = y - \frac{2i}{3}x^{3/2}$, are complex numbers, so the PDE will not transform to the simplest form. Rather, since ξ and η are complex conjugates of each other, we introduce the new real variables,

$$\alpha = \frac{1}{2}(\xi + \eta) = y$$

$$\beta = \frac{1}{2i}(\xi - \eta) = \frac{2}{3}x^{3/2},$$

which do transform the PDE to the simplest form. After changing variables $(x, y) \rightarrow (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_{\alpha} + E^{**}u_{\beta} + F^{**}u = G^{**},$$

where, using the chain rule,

$$A^{**} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2$$

$$B^{**} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y$$

$$C^{**} = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2$$

$$D^{**} = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y$$

$$E^{**} = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y$$

$$F^{**} = F$$

$$G^{**} = G.$$

Plugging in the numbers and derivatives to these equations, we find that $A^{**} = x$, $B^{**} = 0$, $C^{**} = x$, $D^{**} = 0$, $E^{**} = \frac{1}{2\sqrt{x}}$, $F^{**} = 0$, and $G^{**} = 0$. The PDE becomes

$$xu_{\alpha\alpha} + xu_{\beta\beta} + \frac{1}{2\sqrt{x}}u_{\beta} = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2x^{3/2}}u_{\beta}$$

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3\beta}u_{\beta}.$$

This is the canonical form of the elliptic PDE.