Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

(h) $3yu_{xx} - xu_{yy} = 0$

Solution

$3yu_{xx} - xu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 3y$, $B = 0$, $C = -x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{6y}\left(\pm\sqrt{12xy}\right)$$
$$\frac{dy}{dx} = \pm\sqrt{\frac{x}{3y}}.$$

Note that the discriminant, $B^2 - 4AC = 12xy$, can be positive, zero, or negative, depending on whether $xy > 0$, $xy = 0$, or $xy < 0$, respectively. That is,

$$\text{The PDE is} \begin{cases} \text{hyperbolic} & \text{if } xy > 0. \\ \text{parabolic} & \text{if } xy = 0. \\ \text{elliptic} & \text{if } xy < 0. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $(xy > 0)$[1]

The solutions to these ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane. Separating variables and integrating the equations, we find that

$$\frac{2}{3}y^{3/2} = \pm\frac{1}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2} + C_0.$$

The characteristic curves are given by

$$y(x) = \left(\frac{3}{2}C_0 \pm \frac{1}{\sqrt{3}}x^{3/2}\right)^{2/3}.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\frac{1}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2}: \quad \frac{3}{2}C_0 = y^{3/2} + \frac{1}{\sqrt{3}}x^{3/2} = \phi(x, y)$$
$$\text{Working with } +\frac{1}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2}: \quad \frac{3}{2}C_0 = y^{3/2} - \frac{1}{\sqrt{3}}x^{3/2} = \psi(x, y).$$

[1] $xy > 0$ holds in the first and third quadrants.

Now we make the change of variables, $\xi = \phi(x,y) = y^{3/2} + \frac{1}{\sqrt{3}}x^{3/2}$ and $\eta = \psi(x,y) = y^{3/2} - \frac{1}{\sqrt{3}}x^{3/2}$, so that the PDE takes the simplest form. Solving these two equations for x and y gives $x = \left[\frac{\sqrt{3}}{2}(\xi-\eta)\right]^{2/3}$ and $y = \left[\frac{1}{2}(\xi+\eta)\right]^{2/3}$. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $C^* = 0$, $F^* = 0$, $G^* = 0$,

$$\begin{aligned}
B^* &= -9xy = -\frac{9}{2}\left(\frac{3}{2}\right)^{1/3}(\xi-\eta)^{2/3}(\xi+\eta)^{2/3}, \\
D^* &= \frac{3\sqrt{3}}{4}\frac{y}{\sqrt{x}} - \frac{3x}{4\sqrt{y}} = \frac{3\left(\frac{3}{2}\right)^{1/3}\eta}{2(\xi-\eta)^{1/3}(\xi+\eta)^{1/3}}, \\
E^* &= -\frac{3\sqrt{3}}{4}\frac{y}{\sqrt{x}} - \frac{3x}{4\sqrt{y}} = -\frac{3\left(\frac{3}{2}\right)^{1/3}\xi}{2(\xi-\eta)^{1/3}(\xi+\eta)^{1/3}}.
\end{aligned}$$

Thus, the PDE simplifies to

$$-\frac{9}{2}\left(\frac{3}{2}\right)^{1/3}(\xi-\eta)^{2/3}(\xi+\eta)^{2/3}u_{\xi\eta} + \frac{3\left(\frac{3}{2}\right)^{1/3}\eta}{2(\xi-\eta)^{1/3}(\xi+\eta)^{1/3}}u_\xi - \frac{3\left(\frac{3}{2}\right)^{1/3}\xi}{2(\xi-\eta)^{1/3}(\xi+\eta)^{1/3}}u_\eta = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = \frac{\xi}{3(\xi^2-\eta^2)}(\eta u_\xi - \xi u_\eta).$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. Solving these two equations for ξ and η gives $\xi = (\alpha+\beta)/2$ and $\eta = (\alpha-\beta)/2$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{\alpha+\beta}{6\alpha\beta}(\alpha u_\beta - \beta u_\alpha).$$

This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(xy = 0)^2$

Substituting $x = 0$ or $y = 0$ into the PDE reduces it immediately to the canonical form of a parabolic equation, $u_{\eta\eta} = 0$. The characteristic equation reduces to

$$\frac{dy}{dx} = 0.$$

Solving this equation for y gives $y(x) = D$, where D is an arbitrary constant. The characteristic curves in the xy-plane are lines parallel to the x-axis.

Case III: The PDE is elliptic $(xy < 0)^3$

The characteristic equations have no real solutions for $xy < 0$. This means that the two distinct families of characteristic curves lie in the complex plane. Separating variables and integrating the characteristic equations, we find that

$$\frac{dy}{dx} = \pm i\sqrt{\frac{x}{3y}}$$
$$\frac{2}{3}y^{3/2} = \pm\frac{i}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2} + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\frac{i}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2}: \quad \frac{3}{2}C_0 = y^{3/2} + \frac{i}{\sqrt{3}}x^{3/2} = \phi(x, y)$$
$$\text{Working with } +\frac{i}{\sqrt{3}} \cdot \frac{2}{3}x^{3/2}: \quad \frac{3}{2}C_0 = y^{3/2} - \frac{i}{\sqrt{3}}x^{3/2} = \psi(x, y).$$

The variables, $\xi = \phi(x, y) = y^{3/2} + \frac{i}{\sqrt{3}}x^{3/2}$ and $\eta = \psi(x, y) = y^{3/2} - \frac{i}{\sqrt{3}}x^{3/2}$, are complex conjugates of one another, so we introduce the new real variables $\alpha = (\xi + \eta)/2$ and $\beta = (\xi - \eta)/2i$. They are[4]

$$\alpha = \frac{1}{2}(\xi + \eta) = y^{3/2},$$
$$\beta = \frac{1}{2i}(\xi - \eta) = \frac{1}{\sqrt{3}}(-x)^{3/2},$$

After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

[2] $xy = 0$ holds on the x and y axes.

[3] $xy < 0$ holds in the second and fourth quadrants.

[4] Because $xy < 0$, one and only one of the variables needs to have a negative sign when changing variables. Choose $+y$ and $-x$ for this exercise. If this is not done, the canonical form of the elliptic PDE will not be obtained.

where, using the chain rule,

$$\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas gives $A^{**} = -\frac{9xy}{4}$, $B^{**} = 0$, $C^{**} = -\frac{9xy}{4}$, $D^{**} = -\frac{3x}{4\sqrt{y}}$, $E^{**} = \frac{3\sqrt{3}y}{4\sqrt{-x}}$, $F^{**} = 0$, and $G^{**} = 0$. The PDE simplifies to

$$-\frac{9xy}{4}u_{\alpha\alpha} - \frac{9xy}{4}u_{\beta\beta} - \frac{3x}{4\sqrt{y}}u_\alpha + \frac{3\sqrt{3}y}{4\sqrt{-x}}u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3y^{3/2}}u_\alpha + \frac{1}{\sqrt{3}(-x)^{3/2}}u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{3\alpha}u_\alpha + \frac{1}{3\beta}u_\beta = 0.$$

Solving for $u_{\alpha\alpha} + u_{\beta\beta}$ gives

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3}\left(\frac{u_\alpha}{\alpha} + \frac{u_\beta}{\beta}\right).$$

This is the canonical form of the elliptic PDE.