Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

(i) $u_{xx} + 2xu_{xy} + a^2u_{yy} + u = 5$

Solution

$u_{xx} + 2xu_{xy} + a^2u_{yy} + u = 5$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 2x$, $C = a^2$, $D = 0$, $E = 0$, $F = 1$, and $G = 5$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(2x \pm \sqrt{4x^2 - 4a^2}\right)$$
$$\frac{dy}{dx} = x \pm \sqrt{x^2 - a^2}.$$

Note that the discriminant, $B^2 - 4AC = 4x^2 - 4a^2$, can be positive, zero, or negative, depending on whether $x^2 - a^2 > 0$, $x^2 - a^2 = 0$, or $x^2 - a^2 < 0$, respectively. That is,[1]

$$\text{The PDE is} \begin{cases} \text{hyperbolic} & \text{if } |x| > |a|. \\ \text{parabolic} & \text{if } |x| = |a|. \\ \text{elliptic} & \text{if } |x| < |a|. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $(|x| > |a|)$

The solutions to the ordinary differential equations are two real and distinct families of characteristic curves in the xy-plane. Integrating the characteristic equations, we find that

$$y(x) = \frac{1}{2}\left[x\left(x \pm \sqrt{x^2 - a^2}\right) \mp a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right] + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } - \text{ and } +: \quad 2C_0 = 2y - \left[x\left(x - \sqrt{x^2 - a^2}\right) + a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right] = \phi(x, y)$$
$$\text{Working with } + \text{ and } -: \quad 2C_0 = 2y - \left[x\left(x + \sqrt{x^2 - a^2}\right) - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right] = \psi(x, y).$$

Now we make the change of variables,
$\xi = \phi(x, y) = 2y - \left[x\left(x - \sqrt{x^2 - a^2}\right) + a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right]$ and
$\eta = \psi(x, y) = 2y - \left[x\left(x + \sqrt{x^2 - a^2}\right) - a^2 \ln\left(x + \sqrt{x^2 - a^2}\right)\right]$, so that the PDE takes the

[1] Bring a^2 to the right and take the square root of both sides.

simplest form. By eliminating y and solving for x, we obtain the transcendental equation, $\xi - \eta = 2x\sqrt{x^2-a^2} - 2a^2 \ln\left(x + \sqrt{x^2-a^2}\right)$. With the change of variables $(x, y) \to (\xi, \eta)$, the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 16\left(a^2 - x^2\right)$, $C^* = 0$, $D^* = -2 + \frac{2x}{\sqrt{x^2-a^2}}$, $E^* = -2 - \frac{2x}{\sqrt{x^2-a^2}}$, $F^* = 1$, and $G^* = 5$. Thus, the PDE simplifies to

$$16\left(a^2 - x^2\right) u_{\xi\eta} + \left(-2 + \frac{2x}{\sqrt{x^2-a^2}}\right) u_\xi + \left(-2 - \frac{2x}{\sqrt{x^2-a^2}}\right) u_\eta + u = 5$$

$$u_{\xi\eta} = \frac{1}{16(x^2-a^2)}\left[2\left(\frac{x}{\sqrt{x^2-a^2}} - 1\right) u_\xi - 2\left(\frac{x}{\sqrt{x^2-a^2}} + 1\right) u_\eta + u - 5\right].$$

This is the first canonical form of the hyperbolic PDE. Since the transcendental equation cannot be solved for x explicitly, we leave the PDE in terms of x. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{16(x^2-a^2)}\left[2\left(\frac{x}{\sqrt{x^2-a^2}} - 1\right)(u_\alpha + u_\beta) - 2\left(\frac{x}{\sqrt{x^2-a^2}} + 1\right)(u_\alpha - u_\beta) + u - 5\right]$$

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{16(x^2-a^2)}\left(-4u_\alpha + \frac{4x}{x^2-a^2} u_\beta + u - 5\right),$$

where $\beta = 2x\sqrt{x^2-a^2} - 2a^2 \ln\left(x + \sqrt{x^2-a^2}\right)$. This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(|x| = |a|)$

When $x^2 - a^2 = 0$, the equations for the characteristics reduce to

$$\frac{dy}{dx} = x.$$

Integrating this gives the characteristic curves:

$$y(x) = \frac{1}{2}x^2 + C_0.$$

Solving now for the integration constant,

$$C_0 = y - \frac{1}{2}x^2 = \phi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = y - \frac{1}{2}x^2$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. Solving these two equations for x and y gives $x^2 = 2(\eta - \xi)$ and $y = \eta$. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = a^2 - x^2 = 0$, $B^* = 2(a^2 - x^2) = 0$, $C^* = a^2$, $D^* = -1$, $E^* = 0$, $F^* = 1$, and $G^* = 5$. Thus, the PDE simplifies to

$$\begin{aligned}
a^2 u_{\eta\eta} - u_\xi + u &= 5 \\
u_{\eta\eta} &= \frac{1}{a^2}(u_\xi - u + 5).
\end{aligned}$$

This is the canonical form of the parabolic PDE.

Case III: The PDE is elliptic $(x^2 - a^2 < 0)$

The characteristic equations have no real solutions in this case. This means that the two distinct families of characteristic curves lie in the complex plane. Integrating the characteristic equations, we find that

$$\frac{dy}{dx} = x \pm i\sqrt{a^2 - x^2}$$

$$y(x) = \frac{1}{2}\left[x^2 \pm i\left(x\sqrt{a^2 - x^2} + a^2 \tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}\right)\right] + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -i: \quad 2C_0 = 2y - x^2 + i\left(x\sqrt{a^2 - x^2} + a^2 \tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}\right) = \phi(x, y)$$

$$\text{Working with } +i: \quad 2C_0 = 2y - x^2 - i\left(x\sqrt{a^2 - x^2} + a^2 \tan^{-1}\frac{x}{\sqrt{a^2 - x^2}}\right) = \psi(x, y).$$

The variables, $\xi = \phi(x, y) = 2y - x^2 + i\left(x\sqrt{a^2 - x^2} + a^2 \tan^{-1} \frac{x}{\sqrt{a^2-x^2}}\right)$ and $\eta = \psi(x, y) = 2y - x^2 - i\left(x\sqrt{a^2 - x^2} + a^2 \tan^{-1} \frac{x}{\sqrt{a^2-x^2}}\right)$, are complex conjugates of each other, so we introduce the new real variables,

$$\begin{aligned}\alpha &= \frac{1}{2}(\xi + \eta) = 2y - x^2 \\ \beta &= \frac{1}{2i}(\xi - \eta) = x\sqrt{a^2 - x^2} + a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}},\end{aligned}$$

which transform the PDE to

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned}A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\ B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\ C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\ D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\ E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\ F^{**} &= F \\ G^{**} &= G.\end{aligned}$$

Plugging in the numbers and derivatives to these formulas gives $A^{**} = 4\left(a^2 - x^2\right)$, $B^{**} = 0$, $C^{**} = 4\left(a^2 - x^2\right)$, $D^{**} = -2$, $E^{**} = -\frac{2x}{\sqrt{a^2-x^2}}$, $F^{**} = 1$, and $G^{**} = 5$. The PDE becomes

$$4\left(a^2 - x^2\right)u_{\alpha\alpha} + 4\left(a^2 - x^2\right)u_{\beta\beta} - 2u_\alpha - \frac{2x}{\sqrt{a^2 - x^2}}u_\beta + u = 5$$

$$4\left(a^2 - x^2\right)(u_{\alpha\alpha} + u_{\beta\beta}) = 2u_\alpha + \frac{2x}{\sqrt{a^2 - x^2}}u_\beta - u + 5$$

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{4\left(a^2 - x^2\right)}\left(2u_\alpha + \frac{2x}{\sqrt{a^2 - x^2}}u_\beta - u + 5\right).$$

This is the canonical form of the elliptic PDE, where x is defined implicitly in terms of β.