Exercise 1

Classify each of the partial differential equations below as either hyperbolic, parabolic, or elliptic, determine the characteristics, and transform the equations to canonical form:

(j) $y^2u_{xx} + x^2u_{yy} = 0$

Solution

$y^2u_{xx} + x^2u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = y^2$, $B = 0$, $C = x^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)\\ \frac{dy}{dx} &= \frac{1}{2y^2}\left(\pm\sqrt{-4x^2y^2}\right)\\ \frac{dy}{dx} &= \pm\frac{ix}{y}.\end{aligned}$$

Note that $B^2 - 4AC = -4x^2y^2 < 0$, which means that the PDE is **elliptic** for all x and y. Therefore, the solutions to the ordinary differential equations are two distinct families of characteristic curves that lie in the complex plane. Separating variables and integrating the equations, we find that

$$\frac{1}{2}y^2 = \pm\frac{i}{2}x^2 + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned}&\text{Working with } -\frac{i}{2}x^2: \quad 2C_0 = y^2 + ix^2 = \phi(x, y)\\ &\text{Working with } +\frac{i}{2}x^2: \quad 2C_0 = y^2 - ix^2 = \psi(x, y).\end{aligned}$$

The typical variables, $\xi = \phi(x, y) = y^2 + ix^2$ and $\eta = \psi(x, y) = y^2 - ix^2$, are complex numbers, so the PDE will not transform to the simplest form. Rather, since ξ and η are complex conjugates of each other, we introduce the new real variables,

$$\begin{aligned}\alpha &= \frac{1}{2}(\xi + \eta) = y^2\\ \beta &= \frac{1}{2i}(\xi - \eta) = x^2,\end{aligned}$$

which do transform the PDE to the simplest form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these equations, $A^{**} = 4x^2y^2 = 4\alpha\beta$, $B^{**} = 0$, $C^{**} = 4x^2y^2 = 4\alpha\beta$, $D^{**} = 2x^2 = 2\beta$, $E^{**} = 2y^2 = 2\alpha$, $F^{**} = 0$, and $G^{**} = 0$. So the PDE becomes

$$4\alpha\beta u_{\alpha\alpha} + 4\alpha\beta u_{\beta\beta} + 2\beta u_\alpha + 2\alpha u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{1}{2\alpha}u_\alpha + \frac{1}{2\beta}u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2}\left(\frac{u_\alpha}{\alpha} + \frac{u_\beta}{\beta}\right).$$

This is the canonical form of the elliptic PDE.