Exercise 16

Solve the Lamb (1904) problem in geophysics that satisfies the Helmholtz equation in an infinite elastic half-space

$$u_{xx} + u_{zz} + \frac{\omega^2}{c_2^2} u = 0, \qquad -\infty < x < \infty, \quad z > 0,$$

where ω is the frequency and c_2 is the shear wave speed.
At the surface of the half-space ($z = 0$), the boundary condition relating the surface stress to the impulsive point load distribution is given by

$$\mu \frac{\partial u}{\partial z} = -P\delta(x) \quad \text{at } z = 0,$$

where μ is one of the Lamé constants, P is a constant, and

$$u(x, z) \to 0 \quad \text{as } z \to \infty \text{ for } -\infty < x < \infty.$$

Show that the solution in terms of polar coordinates is

$$\begin{aligned} u(x,z) &= \frac{P}{2i\mu} H_0^{(2)}\left(\frac{\omega r}{c_2}\right) \\ &\sim \frac{P}{2i\mu}\left(\frac{2c_2}{\pi\omega r}\right)^{\frac{1}{2}} \exp\left(\frac{\pi i}{4} - \frac{i\omega r}{c_2}\right) \quad \text{for } \omega r \gg c_2. \end{aligned}$$

Solution

The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x,z)\} = U(k,z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,z)\, dx,$$

which means the partial derivatives of u with respect to x and z transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} = (ik)^n U(k,z)$$

$$\mathcal{F}\left\{\frac{\partial^n u}{\partial z^n}\right\} = \frac{d^n U}{dz^n}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\left\{u_{xx} + u_{zz} + \frac{\omega^2}{c_2^2} u\right\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_{xx}\} + \mathcal{F}\{u_{zz}\} + \mathcal{F}\left\{\frac{\omega^2}{c_2^2} u\right\} = 0$$

Transform the derivatives with the relations above.

$$(ik)^2 U + \frac{d^2 U}{dz^2} + \frac{\omega^2}{c_2^2} U = 0$$

Move the terms with U to the right side and factor.

$$\frac{d^2U}{dz^2} = \left(k^2 - \frac{\omega^2}{c_2^2}\right) U$$

The solution to this second-order ODE can be written in terms of exponentials.

$$U(k,z) = A(k)e^{z\sqrt{k^2-\frac{\omega^2}{c_2^2}}} + B(k)e^{-z\sqrt{k^2-\frac{\omega^2}{c_2^2}}}$$

To determine the constants, $A(k)$ and $B(k)$, we have to make use of the boundary conditions. Take the Fourier transform of both sides of them.

$$\begin{aligned} \lim_{z\to\infty} u(x,z) = 0 \quad \to \quad \mathcal{F}\left\{\lim_{z\to\infty} u(x,z)\right\} &= \mathcal{F}\{0\} \\ \lim_{z\to\infty} \mathcal{F}\{u(x,z)\} &= 0 \\ \lim_{z\to\infty} U(k,z) &= 0 \end{aligned} \tag{1}$$

$$\begin{aligned} \mu \frac{\partial u}{\partial z}\bigg|_{z=0} = -P\delta(x) \quad \to \quad \mathcal{F}\left\{\mu \frac{\partial u}{\partial z}\bigg|_{z=0}\right\} &= \mathcal{F}\{-P\delta(x)\} \\ \mu\,\mathcal{F}\left\{\frac{\partial u}{\partial z}\bigg|_{z=0}\right\} &= \frac{-P}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\delta(x)\,dx \\ \mu\,\frac{dU}{dz}\bigg|_{z=0} &= -\frac{P}{\sqrt{2\pi}} \end{aligned} \tag{2}$$

In order for condition (1) to be satisfied, we require that $A(k) = 0$.

$$U(k,z) = B(k)e^{-z\sqrt{k^2-\frac{\omega^2}{c_2^2}}}$$

To make use of condition (2), differentiate $U(k,z)$ with respect to z.

$$\frac{dU}{dz} = -B(k)\sqrt{k^2 - \frac{\omega^2}{c_2^2}}\,e^{-z\sqrt{k^2-\frac{\omega^2}{c_2^2}}}$$

Evaluating this at $z = 0$, we find that

$$\frac{dU}{dz}\bigg|_{z=0} = -B(k)\sqrt{k^2 - \frac{\omega^2}{c_2^2}} = -\frac{P}{\mu\sqrt{2\pi}} \quad \to \quad B(k) = \frac{P}{\mu\sqrt{2\pi}\sqrt{k^2 - \frac{\omega^2}{c_2^2}}}.$$

Hence, the solution for $U(k,z)$ is

$$U(k,z) = \frac{P}{\mu\sqrt{2\pi}\sqrt{k^2 - \frac{\omega^2}{c_2^2}}}e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,z}.$$

To change back to $u(x,z)$, we have to take the inverse Fourier transform of $U(k,z)$. It is defined as

$$u(x,z) = \mathcal{F}^{-1}\{U(k,z)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(k,z)e^{ikz}\,dk.$$

Therefore,

$$u(x,z) = \frac{P}{2\pi\mu}\int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 - \frac{\omega^2}{c_2^2}}}e^{-\sqrt{k^2-\frac{\omega^2}{c_2^2}}\,z}e^{ikx}\,dk.$$