Exercise 17

Find the solution of the Cauchy-Poisson problem (Debnath 1994, p. 83) in inviscid water of infinite depth which is governed by

$$\begin{aligned}
&\phi_{xx} + \phi_{zz} = 0, \quad -\infty < x < \infty,\ -\infty < z \le 0,\ t > 0,\\
&\left.\begin{aligned} \phi_z - \eta_t = 0,\\ \phi_t + g\eta = 0 \end{aligned}\right\} \quad \text{on } z = 0,\ t > 0,\\
&\phi_z \to 0 \quad \text{as } z \to -\infty.\\
&\phi(x, 0, 0) = 0, \quad \text{and} \quad \eta(x, 0) = P\delta(x),
\end{aligned}$$

where $\phi = \phi(x, z, t)$ is the velocity potential, $\eta(x, t)$ is the free surface elevation, and P is a constant.
Derive the asymptotic solution for the free surface elevation as $t \to \infty$.

Solution

The PDEs for ϕ and η are defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve them. We define the Fourier transform with respect to x here as

$$\mathcal{F}_x\{\phi(x, z, t)\} = \Phi(k, z, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \phi(x, z, t)\, dx,$$

which means the partial derivatives of ϕ with respect to x, z, and t transform as follows.

$$\begin{aligned}
\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial x^n}\right\} &= (ik)^n \Phi(k, z, t)\\
\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial z^n}\right\} &= \frac{d^n \Phi}{dz^n}\\
\mathcal{F}_x\left\{\frac{\partial^n \phi}{\partial t^n}\right\} &= \frac{d^n \Phi}{dt^n}
\end{aligned}$$

Take the Fourier transform of both sides of the first PDE.

$$\mathcal{F}_x\{\phi_{xx} + \phi_{zz}\} = \mathcal{F}_x\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}_x\{\phi_{xx}\} + \mathcal{F}_x\{\phi_{zz}\} = 0$$

Transform the derivatives with the relations above.

$$(ik)^2 \Phi + \frac{d^2 \Phi}{dz^2} = 0$$

Expand the coefficient of Φ.

$$-k^2 \Phi + \frac{d^2 \Phi}{dz^2} = 0$$

Bring the term with Φ to the right side.

$$\frac{d^2 \Phi}{dz^2} = k^2 \Phi$$

We can write the solution to this ODE in terms of exponentials.

$$\Phi(k,z,t) = A(k,t)e^{|k|z} + B(k,t)e^{-|k|z}$$

We can determine one of the constants here by using the boundary condition, $\phi_z \to 0$ as $z \to -\infty$. Take the Fourier transform with respect to x of both sides of it.

$$\mathcal{F}_x\left\{\lim_{z\to-\infty}\frac{\partial\phi}{\partial z}\right\} = \mathcal{F}_x\{0\}$$

Bring the Fourier transform inside the limit.

$$\lim_{z\to-\infty}\mathcal{F}_x\left\{\frac{\partial\phi}{\partial z}\right\} = 0$$

Transform the partial derivative.

$$\lim_{z\to-\infty}\frac{d\Phi}{dz} = 0 \tag{1}$$

To use this condition, differentiate $\Phi(k,z,t)$ with respect to z.

$$\frac{d\Phi}{dz} = A(k,t)|k|e^{|k|z} - B(k,t)|k|e^{-|k|z}$$

In order for equation (1) to be satisfied, we require that $B(k,t) = 0$. So we have

$$\Phi(k,z,t) = A(k,t)e^{|k|z}.$$

Take the Fourier transform with respect to x of the boundary conditions now.

$$\begin{aligned}\mathcal{F}_x\{\phi_z - \eta_t\} &= \mathcal{F}_x\{0\}\\ \mathcal{F}_x\{\phi_t + g\eta\} &= \mathcal{F}_x\{0\}\end{aligned}$$

Use the linearity property.

$$\begin{aligned}\mathcal{F}_x\{\phi_z\} - \mathcal{F}_x\{\eta_t\} &= 0\\ \mathcal{F}_x\{\phi_t\} + g\mathcal{F}_x\{\eta\} &= 0\end{aligned}$$

Transform the partial derivatives.

$$\begin{aligned}\frac{d\Phi}{dz} - \frac{dH}{dt} &= 0\\ \frac{d\Phi}{dt} + gH &= 0\end{aligned}$$

Plug in the expression for Φ into these equations. These two equations hold at the boundary, so we have to evaluate these terms at $z = 0$.

$$\begin{aligned}A(k,t)|k| - \frac{dH}{dt} &= 0\\ \frac{\partial A}{\partial t} + gH &= 0\end{aligned} \tag{2}$$

We now have a system of two equations for two unknowns, A and H. Differentiate both sides of the first equation with respect to t.

$$\frac{\partial A}{\partial t}|k| - \frac{d^2H}{dt^2} = 0$$
$$\frac{\partial A}{\partial t} + gH = 0$$

Solve the first equation for A_t

$$\frac{\partial A}{\partial t} = \frac{1}{|k|}\frac{d^2H}{dt^2},$$

and plug it into the second equation.

$$\frac{1}{|k|}\frac{d^2H}{dt^2} + gH = 0$$

Multiply both sides by $|k|$.

$$\frac{d^2H}{dt^2} + g|k|H = 0$$

We can write the solution to this ODE in terms of sine and cosine.

$$H(k,t) = C(k)\cos\sqrt{g|k|}t + D(k)\sin\sqrt{g|k|}t$$

We can determine one of the constants here by using the initial condition, $\eta(x,0) = P\delta(x)$. Take the Fourier transform of both sides of it with respect to x.

$$\mathcal{F}_x\{\eta(x,0)\} = \mathcal{F}_x\{P\delta(x)\}$$
$$H(k,0) = \frac{P}{\sqrt{2\pi}}$$

Using this condition gives us

$$H(k,0) = C(k) = \frac{P}{\sqrt{2\pi}},$$

so we have

$$H(k,t) = \frac{P}{\sqrt{2\pi}}\cos\sqrt{g|k|}t + D(k)\sin\sqrt{g|k|}t.$$

Now we can solve equation (2) for $A(k,t)$.

$$A(k,t)|k| - \frac{dH}{dt} = 0 \quad \rightarrow \quad A(k,t) = \frac{1}{|k|}\frac{dH}{dt}$$

Evaluate the derivative of $H(k,t)$ with respect to t and substitute it.

$$A(k,t) = \frac{1}{|k|}\left[-\frac{P}{\sqrt{2\pi}}\sqrt{g|k|}\sin\sqrt{g|k|}t + D(k)\sqrt{g|k|}\cos\sqrt{g|k|}t\right]$$

We will use the final condition, $\phi(x,0,0) = 0$, now to determine $D(k)$. Take the Fourier transform with respect to x of both sides of it.

$$\mathcal{F}_x\{\phi(x,0,0)\} = \mathcal{F}_x\{0\}$$
$$\Phi(k,0,0) = 0$$

Plug $z = 0$ and $t = 0$ into the expression we found for Φ.

$$A(k, 0) = 0$$

Using this condition, we get

$$A(k, 0) = \frac{1}{|k|}[D(k)\sqrt{g|k|}] = 0 \quad \rightarrow \quad D(k) = 0.$$

Therefore,

$$\Phi(k, z, t) = \frac{1}{|k|}\left[-\frac{P}{\sqrt{2\pi}}\sqrt{g|k|}\sin\sqrt{g|k|}t\right]e^{|k|z}$$
$$H(k, t) = \frac{P}{\sqrt{2\pi}}\cos\sqrt{g|k|}t.$$

All we need to do now is take the inverse Fourier transform of Φ and H, and we'll be done. It is defined as

$$\mathcal{F}^{-1}\{\Phi(k, z, t)\} = \phi(x, z, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi(k, z, t)e^{ikx}\,dk.$$

Plugging Φ and H into the definition, we get

$$\phi(x, z, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{|k|}\left[-\frac{P}{\sqrt{2\pi}}\sqrt{g|k|}\sin\sqrt{g|k|}t\right]e^{|k|z}e^{ikx}\,dk$$
$$\eta(x, t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{P}{\sqrt{2\pi}}\cos\sqrt{g|k|}t\,e^{ikx}\,dk.$$

Bring the constants out in front of the integral to obtain the final answer.

$$\phi(x, z, t) = -\frac{P}{2\pi}\sqrt{g}\int_{-\infty}^{\infty}\frac{\sin\sqrt{g|k|}t}{\sqrt{|k|}}e^{|k|z+ikx}\,dk$$
$$\eta(x, t) = \frac{P}{2\pi}\int_{-\infty}^{\infty}\cos\sqrt{g|k|}t\,e^{ikx}\,dk$$

This answer for ϕ is in disagreement with the answer at the back of the book. $\sqrt{g}$ is in the denominator with 2π there, but I believe this is a typo.