Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(a) $x^2u_{xx} + 4xyu_{xy} + y^2u_{yy} = 0$

Solution

$x^2u_{xx} + 4xyu_{xy} + y^2u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = x^2$, $B = 4xy$, $C = y^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{2x^2}\left(4xy \pm \sqrt{16x^2y^2 - 4x^2y^2}\right) \\
\frac{dy}{dx} &= \frac{1}{2x^2}\left(4xy \pm 2xy\sqrt{3}\right) \\
\frac{dy}{dx} &= \frac{y}{x}\left(2 \pm \sqrt{3}\right).
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 12x^2y^2$, is greater than 0 for all x and y, which means that the PDE is **hyperbolic**. The solutions to the ordinary differential equations are therefore two distinct families of real characteristic curves in the xy-plane. Separating variables and integrating the equations, we find that

$$\ln|y| = \left(2 \pm \sqrt{3}\right)\ln|x| + C_0.$$

Exponentiating both sides gives us the characteristic curves:

$$y(x) = A_0|x|^{\left(2\pm\sqrt{3}\right)}.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned}
\text{Working with } 2 - \sqrt{3}: \quad C_0 &= \ln|y| - \left(2 - \sqrt{3}\right)\ln|x| = \phi(x, y) \\
\text{Working with } 2 + \sqrt{3}: \quad C_0 &= \ln|y| - \left(2 + \sqrt{3}\right)\ln|x| = \psi(x, y).
\end{aligned}$$

Now we make the change of variables, $\xi = \phi(x, y) = \ln|y| - \left(2 - \sqrt{3}\right)\ln|x|$ and
$\eta = \psi(x, y) = \ln|y| - \left(2 + \sqrt{3}\right)\ln|x|$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -12$, $C^* = 0$, $D^* = 1 - \sqrt{3}$, $E^* = 1 + \sqrt{3}$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-12u_{\xi\eta} + \left(1 - \sqrt{3}\right)u_\xi + \left(1 + \sqrt{3}\right)u_\eta = 0.$$

Solving for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = \frac{1}{12}\left[\left(1 - \sqrt{3}\right)u_\xi + \left(1 + \sqrt{3}\right)u_\eta\right].$$

This is the first canonical form of the hyperbolic PDE. To get the second canonical form, we have to make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. The chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{6}(u_\alpha - \sqrt{3}u_\beta).$$

This is the second canonical form of the hyperbolic PDE.