Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(b) $u_{xx} - xu_{yy} = 0$

Solution

$u_{xx} - xu_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = -x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\pm\sqrt{4x}\right)$$
$$\frac{dy}{dx} = \pm\sqrt{x}.$$

Note that the discriminant, $B^2 - 4AC = 4x$, can be positive, zero, or negative, depending on whether $x > 0$, $x = 0$, or $x < 0$, respectively. That is,

$$\text{The PDE is} \begin{cases} \text{hyperbolic} & \text{if } x > 0. \\ \text{parabolic} & \text{if } x = 0. \\ \text{elliptic} & \text{if } x < 0. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $(x > 0)$

The solutions to these ordinary differential equations are two distinct families of real characteristic curves in the xy-plane. Integrating the equations, we find that

$$y(x) = \pm\frac{2}{3}x^{3/2} + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\frac{2}{3}x^{3/2}: \quad C_0 = y + \frac{2}{3}x^{3/2} = \phi(x, y)$$
$$\text{Working with } +\frac{2}{3}x^{3/2}: \quad C_0 = y - \frac{2}{3}x^{3/2} = \psi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = y + \frac{2}{3}x^{3/2}$ and $\eta = \psi(x, y) = y - \frac{2}{3}x^{3/2}$, so that the PDE takes the simplest form. Solving these two equations for x and y gives
$x^{3/2} = \frac{3}{4}(\xi - \eta)$ and $y = \frac{1}{2}(\xi + \eta)$. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -4x$, $C^* = 0$, $D^* = \frac{1}{2\sqrt{x}}$, $E^* = -\frac{1}{2\sqrt{x}}$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-4xu_{\xi\eta} + \frac{1}{2\sqrt{x}}u_\xi - \frac{1}{2\sqrt{x}}u_\eta = 0.$$

Solving now for $u_{\xi\eta}$ gives

$$u_{\xi\eta} = \frac{1}{8x^{3/2}}(u_\xi - u_\eta),$$

which is

$$u_{\xi\eta} = \frac{1}{6(\xi - \eta)}(u_\xi - u_\eta).$$

This is the first canonical form of the hyperbolic PDE. To get the second canonical form, we have to make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. The chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. Changing variables gives us

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{3\beta}u_\beta.$$

This is the second canonical form of the hyperbolic PDE.

Case II: The PDE is parabolic $(x = 0)$

Substituting $x = 0$ into the PDE reduces it immediately to the canonical form of a parabolic equation, $u_{xx} = 0$. The characteristic equations reduce to

$$\frac{dy}{dx} = 0.$$

The characteristic curves in the xy-plane are lines parallel to the x-axis, $y(x) = C_0$, where C_0 is an arbitrary constant.

Case III: The PDE is elliptic $(x < 0)$

The characteristic equations have no real solutions for $x < 0$. This means that the two distinct families of characteristic curves lie in the complex plane. Integrating the characteristic equations, we find that

$$\frac{dy}{dx} = \pm i\sqrt{x}$$

$$y(x) = \pm\frac{2i}{3}x^{3/2} + C_0.$$

Solving for the constant of integration, C_0 (or any convenient multiple thereof),

$$\text{Working with } -\frac{2i}{3}x^{3/2}: \quad C_0 = y + \frac{2i}{3}x^{3/2} = \phi(x, y)$$
$$\text{Working with } +\frac{2i}{3}x^{3/2}: \quad C_0 = y - \frac{2i}{3}x^{3/2} = \psi(x, y).$$

Because $\xi = \phi(x, y) = y + \frac{2i}{3}x^{3/2}$ and $\eta = \psi(x, y) = y - \frac{2i}{3}x^{3/2}$ are complex conjugates of one another, we introduce the new real variables[1],

$$\alpha = \frac{\xi + \eta}{2} = y$$
$$\beta = \frac{\xi - \eta}{2i} = \frac{2}{3}(-x)^{3/2},$$

which transform the PDE to the canonical form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^{**} = -x$, $B^{**} = 0$, $C^{**} = -x$, $D^{**} = 0$, $E^{**} = \frac{1}{2\sqrt{-x}}$, $F^{**} = 0$, and $G^{**} = 0$. Thus, the PDE simplifies to

$$-xu_{\alpha\alpha} - xu_{\beta\beta} + \frac{1}{2\sqrt{-x}}u_\beta = 0$$
$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2(-x)^{3/2}}u_\beta,$$

which is

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{3\beta}u_\beta.$$

This is the canonical form of the elliptic PDE.

[1] Since $x < 0$, we have to use $-x$ in the change of variables. Otherwise, we will not get the desired canonical form.