Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(c) $u_{xx} - 2u_{xy} + 3u_{yy} + 24u_y + 5u = 0$

Solution

$u_{xx} - 2u_{xy} + 3u_{yy} + 24u_y + 5u = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = -2$, $C = 3$, $D = 0$, $E = 24$, $F = 5$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(-2 \pm \sqrt{4 - 12}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(-2 \pm 2i\sqrt{2}\right)$$
$$\frac{dy}{dx} = -1 \pm i\sqrt{2}.$$

Note that the discriminant, $B^2 - 4AC = 4 - 12 = -8$, is less than 0, which means that the PDE is **elliptic** for all x and y . Therefore, the solutions to the ordinary differential equations are two distinct families of characteristic curves that lie in the complex plane. Integrating the characteristic equations, we find that

$$y(x) = \left(-1 \pm i\sqrt{2}\right)x + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -i\sqrt{2}: \quad C_0 = y + x + ix\sqrt{2} = \phi(x, y)$$
$$\text{Working with } +i\sqrt{2}: \quad C_0 = y + x - ix\sqrt{2} = \psi(x, y).$$

Since $\xi = \phi(x, y) = y + x + ix\sqrt{2}$ and $\eta = \psi(x, y) = y + x - ix\sqrt{2}$ are complex conjugates of one another, we introduce the new real variables,

$$\alpha = \frac{\xi + \eta}{2} = y + x$$
$$\beta = \frac{\xi - \eta}{2i} = x\sqrt{2},$$

which transform the PDE to the canonical form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_{\alpha} + E^{**}u_{\beta} + F^{**}u = G^{**},$$

where, using the chain rule,

$$
\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}
$$

Plugging in the numbers and derivatives to these equations, we find that $A^{**} = 2$, $B^{**} = 0$, $C^{**} = 2$, $D^{**} = 24$, $E^{**} = 0$, $F^{**} = 5$, and $G^{**} = 0$. Thus, the PDE simplifies to

$$2u_{\alpha\alpha} + 2u_{\beta\beta} + 24u_\alpha + 5u = 0.$$

Solving for $u_{\alpha\alpha} + u_{\beta\beta}$ gives

$$u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2}(24u_\alpha + 5u).$$

This is the canonical form of the elliptic PDE.