Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(d) $u_{xx} + \operatorname{sech}^4 x u_{yy} = 0$

Solution

$u_{xx} + \operatorname{sech}^4 x u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = \operatorname{sech}^4 x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\pm\sqrt{-4\operatorname{sech}^4 x}\right)$$
$$\frac{dy}{dx} = \pm i \operatorname{sech}^2 x.$$

Note that the discriminant, $B^2 - 4AC = -4\operatorname{sech}^4 x$, is less than 0 for all x, which means that the PDE is **elliptic**. Therefore, the solutions to the ordinary differential equations are two distinct families of characteristic curves that lie in the complex plane. Integrating the characteristic equations, we find that

$$y(x) = \pm i \tanh x + C_0$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -i\tanh x\text{:} \quad C_0 = y + i\tanh x = \phi(x, y)$$
$$\text{Working with } +i\tanh x\text{:} \quad C_0 = y - i\tanh x = \psi(x, y).$$

Since $\xi = \phi(x, y) = y + i\tanh x$ and $\eta = \psi(x, y) = y - i\tanh x$ are complex conjugates of one another, we introduce the new real variables,

$$\alpha = \frac{1}{2}(\xi + \eta) = y$$
$$\beta = \frac{1}{2i}(\xi - \eta) = \tanh x,$$

which transform the PDE to the canonical form. After changing variables $(x, y) \to (\alpha, \beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_\alpha + E^{**}u_\beta + F^{**}u = G^{**},$$

where, using the chain rule,

$$A^{**} = A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2$$
$$B^{**} = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y$$
$$C^{**} = A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2$$
$$D^{**} = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y$$
$$E^{**} = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y$$
$$F^{**} = F$$
$$G^{**} = G.$$

Plugging in the numbers and derivatives to these formulas, we find that
$A^{**} = \operatorname{sech}^4 x = \left(\beta^2 - 1\right)^2$, $B^{**} = 0$, $C^{**} = \operatorname{sech}^4 x = \left(\beta^2 - 1\right)^2$, $D^{**} = 0$,
$E^{**} = -2\operatorname{sech}^2 x \tanh x = 2\beta\left(\beta^2 - 1\right)$, $F^{**} = 0$, and $G^{**} = 0$. Thus, the PDE simplifies to

$$\left(\beta^2 - 1\right)^2 u_{\alpha\alpha} + \left(\beta^2 - 1\right)^2 u_{\beta\beta} + 2\beta\left(\beta^2 - 1\right) u_\beta = 0$$

$$u_{\alpha\alpha} + u_{\beta\beta} + \frac{2\beta}{\beta^2 - 1} u_\beta = 0.$$

Solving for $u_{\alpha\alpha} + u_{\beta\beta}$ gives

$$u_{\alpha\alpha} + u_{\beta\beta} = \frac{2\beta}{1 - \beta^2} u_\beta.$$

This is the canonical form of the elliptic PDE.