Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(e) $u_{xx} + 6yu_{xy} + 9y^2u_{yy} + 4u = 0$

Solution

$u_{xx} + 6yu_{xy} + 9y^2u_{yy} + 4u = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 6y$, $C = 9y^2$, $D = 0$, $E = 0$, $F = 4$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(6y \pm \sqrt{36y^2 - 36y^2}\right)$$
$$\frac{dy}{dx} = 3y.$$

Note that the discriminant, $B^2 - 4AC = 36y^2 - 36y^2$, is equal to 0 for all y, which means that the PDE is **parabolic**. Therefore, there is one family of real characteristic curves in the xy-plane. Integrating the characteristic equation, we find that

$$\ln|y| = 3x + C_0.$$

So the characteristic curves are given by

$$y(x) = A_0e^{3x}.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$C_0 = \ln|y| - 3x = \phi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = \ln|y| - 3x$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$
$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$
$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$
$$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$
$$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$
$$F^* = F$$
$$G^* = G.$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 0$, $C^* = 9y^2 = 9\eta^2$, $D^* = -9$, $E^* = 0$, $F^* = 4$, and $G^* = 0$. Thus, the PDE simplifies to

$$9\eta^2 u_{\eta\eta} - 9u_\eta + 4u = 0$$
$$u_{\eta\eta} = \frac{1}{9\eta^2}(9u_\eta - 4u).$$

This is the canonical form of the parabolic PDE.