Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(f) $u_{xx} - \operatorname{sech}^4 x u_{yy} = 0$

Solution

$u_{xx} - \operatorname{sech}^4 x u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 0$, $C = -\operatorname{sech}^4 x$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\ \frac{dy}{dx} &= \frac{1}{2}\left(\pm\sqrt{4\operatorname{sech}^4 x}\right) \\ \frac{dy}{dx} &= \pm \operatorname{sech}^2 x. \end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4\operatorname{sech}^4 x$, is greater than 0 for all x, which means that the PDE is **hyperbolic**. The two families of characteristic curves, therefore, are distinct and lie in the xy-plane. Integrating the characteristic equations, we find that

$$y(x) = \pm \tanh x + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\begin{aligned} &\text{Working with } -\tanh x: \quad C_0 = y + \tanh x = \phi(x, y) \\ &\text{Working with } +\tanh x: \quad C_0 = y - \tanh x = \psi(x, y). \end{aligned}$$

Make the change of variables, $\xi = \phi(x, y) = y + \tanh x$ and $\eta = \psi(x, y) = y - \tanh x$, so that the PDE takes the simplest form. Solving these two equations for x and y gives $x = \tanh^{-1}\left[\frac{1}{2}(\xi - \eta)\right]$ and $y = \frac{1}{2}(\xi + \eta)$. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned} A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\ D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\ E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\ F^* &= F \\ G^* &= G. \end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $C^* = 0$, $F^* = 0$, $G^* = 0$,

$$\begin{aligned} B^* &= -4\operatorname{sech}^4 x = -\frac{1}{4}\left[(\xi-\eta)^2 - 4\right]^2, \\ D^* &= -2\operatorname{sech}^2 x \tanh x = \frac{1}{4}(\xi-\eta)\left[(\xi-\eta)^2 - 4\right], \\ E^* &= 2\operatorname{sech}^2 x \tanh x = -\frac{1}{4}(\xi-\eta)\left[(\xi-\eta)^2 - 4\right]. \end{aligned}$$

Thus, the PDE simplifies to

$$-\frac{1}{4}\left[(\xi-\eta)^2 - 4\right]^2 u_{\xi\eta} + \frac{1}{4}(\xi-\eta)\left[(\xi-\eta)^2 - 4\right](u_\xi - u_\eta) = 0$$

$$u_{\xi\eta} + \frac{\xi-\eta}{4-(\xi-\eta)^2}(u_\xi - u_\eta) = 0$$

$$u_{\xi\eta} = \frac{\xi-\eta}{(\xi-\eta)^2-4}(u_\xi - u_\eta).$$

This is the first canonical form of the hyperbolic PDE. To get the second canonical form, we have to make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$. The chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{2\beta}{\beta^2 - 4} u_\beta.$$

This is the second canonical form of the hyperbolic PDE.