Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(g) $u_{xx} + 2\operatorname{cosec} y u_{xy} + \operatorname{cosec}^2 y u_{yy} = 0$

Solution

$u_{xx} + 2\operatorname{cosec} y u_{xy} + \operatorname{cosec}^2 y u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = 2\operatorname{cosec} y$, $C = \operatorname{cosec}^2 y$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right) \\
\frac{dy}{dx} &= \frac{1}{2}\left(2\operatorname{cosec} y \pm \sqrt{4\operatorname{cosec}^2 y - 4\operatorname{cosec}^2 y}\right) \\
\frac{dy}{dx} &= \operatorname{cosec} y = \frac{1}{\sin y}.
\end{aligned}$$

Note that the discriminant, $B^2 - 4AC = 4\operatorname{cosec}^2 y - 4\operatorname{cosec}^2 y$, is equal to 0 for all y, which means that the PDE is **parabolic**. Therefore, there is one family of characteristic curves in the xy-plane. Separating variables and integrating the characteristic equation, we find that

$$-\cos y = x + C_0,$$

and the characteristic curves are given by

$$y(x) = \cos^{-1}(-x - C_0).$$

Solving for the constant of integration (or any convenient multiple thereof),

$$-C_0 = x + \cos y = \phi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = x + \cos y$. η can be chosen arbitrarily so long as the Jacobian of ξ and η is nonzero. We choose $\eta = y$ for simplicity. With these new variables the PDE becomes

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
F^* &= F \\
G^* &= G.
\end{aligned}$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = 0$, $C^* = \operatorname{cosec}^2 y = \operatorname{cosec}^2 \eta$, $D^* = -\cot y \operatorname{cosec} y = -\cot \eta \operatorname{cosec} \eta$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$(\operatorname{cosec}^2 \eta) u_{\eta\eta} - (\cot \eta \operatorname{cosec} \eta) u_\xi = 0$$

$$\frac{1}{\sin^2 \eta} u_{\eta\eta} - \frac{\cos \eta}{\sin \eta} \frac{1}{\sin \eta} u_\xi = 0$$

$$u_{\eta\eta} = \frac{\cot \eta}{\operatorname{cosec} \eta} u_\xi$$

$$u_{\eta\eta} = (\cos \eta) u_\xi.$$

This is the canonical form of the parabolic PDE.

This answer is in disagreement with the answer at the back of the book–there is an extra $\sin^2 \eta$ term on the right. I believe the book is in error.