Exercise 2

Determine the nature of the following equations and reduce them to canonical form:

(h) $u_{xx} - 5u_{xy} + 5u_{yy} = 0$

Solution

$u_{xx} - 5u_{xy} + 5u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = -5$, $C = 5$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. The characteristic equations of this PDE are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(-5 \pm \sqrt{25 - 20}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(-5 \pm \sqrt{5}\right).$$

Note that the discriminant, $B^2 - 4AC = 25 - 20$, is equal to 5, which means that the PDE is **hyperbolic**. The two families of characteristic curves, therefore, are distinct and lie in the xy-plane. Integrating the characteristic equations, we find that

$$y(x) = \frac{1}{2}\left(-5 \pm \sqrt{5}\right)x + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -\sqrt{5}: \quad 2C_0 = 2y + \left(5 + \sqrt{5}\right)x = \phi(x, y)$$
$$\text{Working with } +\sqrt{5}: \quad 2C_0 = 2y + \left(5 - \sqrt{5}\right)x = \psi(x, y).$$

Now we make the change of variables, $\xi = \phi(x, y) = 2y + \left(5 + \sqrt{5}\right)x$ and
$\eta = \psi(x, y) = 2y + \left(5 - \sqrt{5}\right)x$, so that the PDE takes the simplest form. With these new variables the PDE becomes

$$A^*u_{\xi\xi} + B^*u_{\xi\eta} + C^*u_{\eta\eta} + D^*u_\xi + E^*u_\eta + F^*u = G^*,$$

where, using the chain rule, (see page 11 of the textbook for details)

$$A^* = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$$
$$B^* = 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y$$
$$C^* = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$$
$$D^* = A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y$$
$$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$$
$$F^* = F$$
$$G^* = G.$$

Plugging in the numbers and derivatives to these formulas, we find that $A^* = 0$, $B^* = -20$, $C^* = 0$, $D^* = 0$, $E^* = 0$, $F^* = 0$, and $G^* = 0$. Thus, the PDE simplifies to

$$-20u_{\xi\eta} = 0$$
$$u_{\xi\eta} = 0.$$

This is the first canonical form of the hyperbolic PDE. If we make the additional change of variables, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, then the chain rule gives $u_{\xi\eta} = u_{\alpha\alpha} - u_{\beta\beta}$, $u_\xi = u_\alpha + u_\beta$, and $u_\eta = u_\alpha - u_\beta$. The PDE then becomes

$$u_{\alpha\alpha} - u_{\beta\beta} = 0.$$

This is the second canonical form of the hyperbolic PDE.