Exercise 24

The transverse vibration of a thin membrane of great extent satisfies the wave equation

$$c^2(u_{xx} + u_{yy}) = u_{tt}, \quad -\infty < x, y < \infty,\ t > 0,$$

with the initial and boundary conditions

$$\begin{aligned} u(x, y, t) &\to 0 \quad \text{as } |x| \to \infty,\ |y| \to \infty \text{ for all } t \geq 0, \\ u(x, y, 0) &= f(x, y), \quad u_t(x, y, 0) = 0 \quad \text{for all } x, y. \end{aligned}$$

Apply the double Fourier transform method to solve this problem.

Solution

The PDE is defined for $-\infty < x, y < \infty$, so we can apply the double Fourier transform to solve it. We define the double Fourier transform here as

$$\mathcal{F}\{u(x, y, t)\} = U(k_1, k_2, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(k_1 x + k_2 y)} u(x, y, t)\, dx\, dy,$$

which means the partial derivatives of u with respect to x, y, and t transform as follows.

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} &= (ik_1)^n U(k_1, k_2, t) \\ \mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} &= (ik_2)^n U(k_1, k_2, t) \\ \mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} &= \frac{d^n U}{dt^n} \end{aligned}$$

Take the double Fourier transform of both sides of the PDE.

$$\mathcal{F}\{c^2(u_{xx} + u_{yy})\} = \mathcal{F}\{u_{tt}\}$$

The double Fourier transform is a linear operator.

$$c^2 \mathcal{F}\{u_{xx}\} + c^2 \mathcal{F}\{u_{yy}\} = \mathcal{F}\{u_{tt}\}$$

Transform the derivatives with the relations above.

$$c^2(ik_1)^2 U + c^2(ik_2)^2 U = \frac{d^2 U}{dt^2}$$

Expand the coefficients of U and factor c^2.

$$\frac{d^2 U}{dt^2} = -c^2(k_1^2 + k_2^2) U \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. Taking the Fourier transform of the initial conditions gives

$$\begin{aligned} u(x, y, 0) = f(x, y) \qquad &\to \qquad \mathcal{F}\{u(x, y, 0)\} = \mathcal{F}\{f(x, y)\} \\ & \qquad\qquad U(k_1, k_2, 0) = F(k_1, k_2) \qquad (2) \\ \frac{\partial u}{\partial t}(x, y, 0) = 0 \qquad &\to \qquad \mathcal{F}\left\{\frac{\partial u}{\partial t}(x, y, 0)\right\} = \mathcal{F}\{0\} \\ & \qquad\qquad \frac{dU}{dt}(k_1, k_2, 0) = 0. \qquad (3) \end{aligned}$$

Equation (1) is an ODE in t, so k_1 and k_2 are treated as constants. The solution to the ODE is given in terms of sine and cosine.

$$U(k_1, k_2, t) = A(k_1, k_2)\cos\left(ct\sqrt{k_1^2 + k_2^2}\right) + B(k_1, k_2)\sin\left(ct\sqrt{k_1^2 + k_2^2}\right)$$

Apply the first initial condition, equation (2).

$$U(k_1, k_2, 0) = A(k_1, k_2) = F(k_1, k_2)$$

In order to apply the second initial condition, differentiate $U(k_1, k_2, t)$ with respect to t.

$$\frac{dU}{dt}(k_1, k_2, t) = -cA(k_1, k_2)\sqrt{k_1^2 + k_2^2}\sin\left(ct\sqrt{k_1^2 + k_2^2}\right) + cB(k_1, k_2)\sqrt{k_1^2 + k_2^2}\cos\left(ct\sqrt{k_1^2 + k_2^2}\right)$$

Now apply equation (3).

$$\frac{dU}{dt}(k_1, k_2, 0) = cB(k_1, k_2)\sqrt{k_1^2 + k_2^2} = 0 \quad \rightarrow \quad B(k_1, k_2) = 0$$

Therefore, the solution to the ODE that satisfies the initial conditions is

$$U(k_1, k_2, t) = F(k_1, k_2)\cos\left(ct\sqrt{k_1^2 + k_2^2}\right).$$

Now that we have $U(k_1, k_2, t)$, we can get $u(x, y, t)$ by taking the inverse Fourier transform of it. It is defined as

$$\mathcal{F}^{-1}\{U(k_1, k_2, t)\} = u(x, y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(k_1, k_2, t)e^{i(k_1x + k_2y)}\, dk_1\, dk_2.$$

Therefore,

$$u(x, y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_1, k_2)\cos\left(ct\sqrt{k_1^2 + k_2^2}\right)e^{i(k_1x + k_2y)}\, dk_1\, dk_2.$$