Exercise 25

Solve the diffusion problem with a source function $q(x,t)$

$$\begin{aligned} u_t &= \kappa u_{xx} + q(x,t), \quad -\infty < x < \infty,\ t > 0, \\ u(x,0) &= 0 \quad \text{for } -\infty < x < \infty. \end{aligned}$$

Show that the solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa}} \int_0^t (t-\tau)^{-\frac{1}{2}}\, d\tau \int_{-\infty}^{\infty} q(k,\tau) \exp\left[-\frac{(x-k)^2}{4\kappa(t-\tau)}\right] dk.$$