Exercise 26

Apply the triple Fourier transform to solve the initial-value problem

$$\begin{aligned} u_t &= \kappa(u_{xx} + u_{yy} + u_{zz}), \quad -\infty < x, y, z < \infty,\ t > 0, \\ u(\mathbf{x}, 0) &= f(\mathbf{x}) \quad \text{for all } x, y, z, \end{aligned}$$

where $\mathbf{x} = (x, y, z)$.

Solution

The PDE is defined for $-\infty < x, y, z < \infty$, so we can apply the triple Fourier transform to solve it. We define the triple Fourier transform here as

$$\mathcal{F}\{u(x,y,z,t)\} = U(k_1,k_2,k_3,t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(k_1x + k_2y + k_3z)} u(x,y,z,t)\, dx\, dy\, dz,$$

which means the partial derivatives of u with respect to x, y, z, and t transform as follows.

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} &= (ik_1)^n U(k_1,k_2,k_3,t) \\ \mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} &= (ik_2)^n U(k_1,k_2,k_3,t) \\ \mathcal{F}\left\{\frac{\partial^n u}{\partial y^n}\right\} &= (ik_3)^n U(k_1,k_2,k_3,t) \\ \mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} &= \frac{d^n U}{dt^n} \end{aligned}$$

Take the triple Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_t\} = \mathcal{F}\{\kappa(u_{xx} + u_{yy} + u_{zz})\}$$

The triple Fourier transform is a linear operator.

$$\mathcal{F}\{u_t\} = \kappa\mathcal{F}\{u_{xx}\} + \kappa\mathcal{F}\{u_{yy}\} + \kappa\mathcal{F}\{u_{zz}\}$$

Transform the derivatives with the relations above.

$$\frac{dU}{dt} = \kappa(ik_1)^2 U + \kappa(ik_2)^2 U + \kappa(ik_3)^2 U$$

Expand the coefficients of U and factor $-\kappa$.

$$\frac{dU}{dt} = -\kappa(k_1^2 + k_2^2 + k_3^2)U \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial condition as well. Taking the Fourier transform of the initial condition gives

$$\begin{aligned} u(x,y,z,0) = f(x,y,z) \qquad \rightarrow \qquad \mathcal{F}\{u(x,y,z,0)\} &= \mathcal{F}\{f(x,y,z)\} \\ U(k_1,k_2,k_3,0) &= F(k_1,k_2,k_3) \qquad (2) \end{aligned}$$

Equation (1) is an ODE in t, so k_1 and k_2 and k_3 are treated as constants. It can be solved with the method of separation of variables.

$$\frac{dU}{U} = -\kappa(k_1^2 + k_2^2 + k_3^2)\,dt$$

Integrate both sides.

$$\ln|U| = -\kappa(k_1^2 + k_2^2 + k_3^2)t + C$$

Exponentiate both sides.

$$|U| = e^{-\kappa(k_1^2+k_2^2+k_3^2)t}e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$U(k_1, k_2, k_3, t) = \pm e^C e^{-\kappa(k_1^2+k_2^2+k_3^2)t}$$

Use a new constant A.

$$U(k_1, k_2, k_3, t) = A(k_1, k_2, k_3)e^{-\kappa(k_1^2+k_2^2+k_3^2)t}$$

Apply the initial condition, equation (2).

$$U(k_1, k_2, k_3, 0) = A(k_1, k_2, k_3) = F(k_1, k_2, k_3)$$

Therefore, the solution to the ODE that satisfies the initial condition is

$$U(k_1, k_2, k_3, t) = F(k_1, k_2, k_3)e^{-\kappa(k_1^2+k_2^2+k_3^2)t}.$$

Now that we have $U(k_1, k_2, k_3, t)$, we can get $u(x, y, z, t)$ by taking the inverse Fourier transform of it.

$$\begin{aligned} u(x, y, z, t) &= \mathcal{F}^{-1}\{U(k_1, k_2, k_3, t)\} \\ &= \mathcal{F}^{-1}\{F(k_1, k_2, k_3)e^{-\kappa(k_1^2+k_2^2+k_3^2)t}\} \end{aligned}$$

Because we are taking the inverse Fourier transform of a product of two functions, $F(k_1, k_2, k_3)$ and $e^{-\kappa(k_1^2+k_2^2+k_3^2)t}$, we can apply the convolution theorem, which states that

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}-\mathbf{x}')g(\mathbf{x}')\,d\mathbf{x}' = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\mathbf{x}')g(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}'.$$

Before we can use it, though, we have to find the inverse Fourier transform of the exponential function. Plug it into the definition of the inverse Fourier transform.

$$\mathcal{F}^{-1}\{e^{-\kappa(k_1^2+k_2^2+k_3^2)t}\} = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\kappa(k_1^2+k_2^2+k_3^2)t}e^{i(k_1x+k_2y+k_3z)}\,dk_1\,dk_2\,dk_3$$

Group the exponentials like so.

$$\mathcal{F}^{-1}\{e^{-\kappa(k_1^2+k_2^2+k_3^2)t}\} = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\kappa k_1^2t+ik_1x}e^{-\kappa k_2^2t+ik_2y}e^{-\kappa k_3^2t+ik_3z}\,dk_1\,dk_2\,dk_3$$

Split up the triple integral into three single integrals.

$$\mathcal{F}^{-1}\{e^{-\kappa(k_1^2+k_2^2+k_3^2)t}\} = \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\kappa k_1^2t+ik_1x}\,dk_1\right)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\kappa k_2^2t+ik_2y}\,dk_2\right)\times \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\kappa k_3^2t+ik_3z}\,dk_3\right)$$

Each of these integrals is similar; if we can determine the first one, then the others are found as well by replacing x with y and z.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\kappa k_1^2 t + ik_1 x}\,dk_1$$

Complete the square in the exponent.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\kappa t\left(k_1 + \frac{ix}{2\kappa t}\right)^2 + \frac{i^2x^2}{4\kappa t}}$$

Distribute κt and split up the exponentials.

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{\kappa t}k_1 + \frac{ix}{2\sqrt{\kappa t}}\right)^2} e^{-\frac{x^2}{4\kappa t}}\,dk_1$$

Make a v-substitution.

$$v = \sqrt{\kappa t}k_1 + \frac{ix}{2\sqrt{\kappa t}}$$
$$dv = \sqrt{\kappa t}\,dk_1 \quad \to \quad \frac{dv}{\sqrt{\kappa t}}$$

The integral becomes

$$\frac{e^{-\frac{x^2}{4\kappa t}}}{\sqrt{2\pi\kappa t}}\int_{-\infty}^{\infty} e^{-v^2}\,dv,$$

which is known to be $\sqrt{\pi}$.

$$\frac{e^{-\frac{x^2}{4\kappa t}}}{\sqrt{2\pi\kappa t}}\cdot\sqrt{\pi} = \frac{1}{\sqrt{2\kappa t}}e^{-\frac{x^2}{4\kappa t}}$$

Thus, we have the inverse Fourier transform of the exponential function.

$$\mathcal{F}^{-1}\{e^{-\kappa(k_1^2+k_2^2+k_3^2)t}\} = \frac{1}{(2\kappa t)^{3/2}}e^{-\frac{x^2}{4\kappa t}}e^{-\frac{y^2}{4\kappa t}}e^{-\frac{z^2}{4\kappa t}}$$

Then by the convolution theorem,

$$u(x,y,z,t) = \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y',z')\frac{1}{(2\kappa t)^{3/2}}e^{-\frac{(x-x')^2}{4\kappa t}}e^{-\frac{(y-y')^2}{4\kappa t}}e^{-\frac{(z-z')^2}{4\kappa t}}\,dx'\,dy'\,dz'.$$

Therefore,

$$u(x,y,z,t) = \frac{1}{(4\pi\kappa t)^{3/2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y',z')e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\kappa t}}\,dx'\,dy'\,dz'.$$