Exercise 28

Use the Fourier transform to solve the Rossby wave problem in an inviscid β-plane ocean bounded by walls at $y = 0$ and $y = 1$, where y and x represent vertical and horizontal directions. The fluid is initially at rest and then, at $t = 0+$, an arbitrary disturbance localized to the vicinity of $x = 0$ is applied to generate Rossby waves. This problem satisfies the Rossby wave equation

$$\frac{\partial}{\partial t}[(\nabla^2 - \kappa^2)\psi] + \beta\psi_x = 0, \quad -\infty < x < \infty,\ 0 \le y \le 1,\ t > 0,$$

with the boundary and initial conditions

$$\begin{aligned} \psi_x(x, y) &= 0 \quad \text{for } 0 < x < \infty,\ y = 0 \text{ and } y = 1, \\ \psi(x, y, t) &= \psi_0(x, y) \quad \text{at } t = 0 \text{ for all } x \text{ and } y. \end{aligned}$$

Examine the case for $\psi_{0n}(x) = \frac{1}{\alpha\sqrt{2}} \exp\{ik_0 x - (\frac{x}{a})^2\}$.

Solution

For cartesian coordinates in two dimensions, the laplacian operator is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Substituting this into the PDE gives

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \kappa^2\right)\psi\right] + \beta\psi_x = 0.$$

Distribute the operator in the square brackets.

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \kappa^2\psi\right) + \beta\psi_x = 0.$$

Distribute the t-derivative.

$$\psi_{xxt} + \psi_{yyt} - \kappa^2\psi_t + \beta\psi_x = 0$$

The PDE is linear and homogeneous, so the method of separation of variables can be used to solve it. The boundary conditions at $y = 0$ and $y = 1$ suggest a solution of the form: $\psi(x, y, t) = A(x, t)Y(y)$. Plugging this form into the boundary conditions, we get

$$\psi_x(x, 0) = A_x(x, t)Y(0) = 0 \quad \rightarrow \quad Y(0) = 0 \tag{1}$$

$$\psi_x(x, 1) = A_x(x, t)Y(1) = 0 \quad \rightarrow \quad Y(1) = 0. \tag{2}$$

Plugging the form into the PDE, we obtain

$$A_{xxt}Y + A_tY'' - \kappa^2A_tY + \beta A_xY = 0.$$

Divide both sides by Y and solve for Y''/Y.

$$\frac{Y''}{Y} = \frac{\kappa^2A_t - \beta A_x - A_{xxt}}{A_t}$$

The left side is a function of y, and the right side is a function of x and t. As these are independent variables, the only way both sides can be equal is if they are both a constant. This constant has to be negative so that the resulting ODE for Y yields a nontrivial solution.

$$\frac{Y''}{Y} = \frac{\kappa^2 A_t - \beta A_x - A_{xxt}}{A_t} = -\lambda^2$$

The Rossby wave equation has thus been reduced to an ODE and a PDE in only two variables, x and t.

$$Y'' = -\lambda^2 Y \quad \text{and} \quad \kappa^2 A_t - \beta A_x - A_{xxt} = -\lambda^2 A_t$$

The solution for the ODE can be written in terms of sine and cosine.

$$Y(y) = C_1 \cos \lambda y + C_2 \sin \lambda y$$

We can use equations (1) and (2) to determine C_1 and C_2. Applying equation (1), we have

$$Y(0) = C_1 = 0.$$

Applying equation (2), we have

$$Y(1) = C_2 \sin \lambda = 0.$$

In order to obtain a nontrivial solution, C_2 cannot be zero. Dividing both sides by C_2 gives

$$\sin \lambda = 0,$$

which means that

$$\lambda = n\pi,$$

where $n = 1, 2, \ldots$. These are the eigenvalues, the values of the constant for which the ODE and boundary conditions are satisfied. The solution to the ODE, also known as the eigenfunctions, are $Y_n(y) = \sin n\pi y$. Only positive values for n are considered because negative values only change the sign, not the magnitude, and $n = 0$ yields the trivial solution. Let's turn our attention now to the PDE.

$$\kappa^2 A_t - \beta A_x - A_{xxt} = -\lambda^2 A_t$$

Plug in $\lambda = n\pi$, bring all terms to the right side, and factor A_t.

$$A_{xxt} + \beta A_x - [\kappa^2 + (n\pi)^2] A_t = 0$$

Since $-\infty < x < \infty$, we can solve this PDE with the Fourier transform. We define it here as

$$\mathcal{F}\{A(x,t)\} = \overline{A}(k,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} A(x,t)\, dx,$$

which means the partial derivatives of A with respect to x and t transform as follows.

$$\mathcal{F}\left\{\frac{\partial^n A}{\partial x^n}\right\} = (ik)^n \overline{A}(k,t)$$

$$\mathcal{F}\left\{\frac{\partial^n A}{\partial t^n}\right\} = \frac{d^n \overline{A}}{dt^n}$$

Take the Fourier transform of both sides of the PDE.

$$(ik)^2 \frac{d\overline{A}}{dt} + \beta(ik)\overline{A} - [\kappa^2 + (n\pi)^2] \frac{d\overline{A}}{dt} = 0$$

Factor $d\overline{A}/dt$, change i^2 to -1, and bring the term with $\overline{A}$ to the other side.

$$[k^2+\kappa^2+(n\pi)^2]\frac{d\overline{A}}{dt} = ik\beta\overline{A}$$

This is a first-order ODE in t that can be solved with separation of variables.

$$\frac{d\overline{A}}{\overline{A}} = \frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}\,dt$$

Integrate both sides.

$$\ln|\overline{A}| = \frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t + C_3(k)$$

Exponentiate both sides.

$$|\overline{A}| = e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t + C_3(k)}$$

Introduce $\pm$ on the right side to remove the absolute value sign on the left.

$$\overline{A}(k,t) = \pm e^{C_3(k)} e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}$$

Use a new arbitrary constant.

$$\overline{A}_n(k,t) = B_n(k) e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}$$

The solution to the Rossby wave equation is obtained by summing all the eigenfunctions together for every value of n. This is the principle of linear superposition.

$$\psi(x,y,t) = \sum_{n=1}^{\infty} A_n(x,t)Y_n(y)$$

Our aim now is to determine $B_n(k)$ with the provided initial condition, $\psi(x,y,0) = \psi_0(x,y)$. Take the Fourier transform of both sides of it.

$$\psi(x,y,0) = \psi_0(x,y) \quad \rightarrow \quad \mathcal{F}\{\psi(x,y,0)\} = \mathcal{F}\{\psi_0(x,y)\}$$
$$\Psi(k,y,0) = \Psi_0(k,y) \tag{3}$$

Now take the Fourier transform of $\psi(x,y,t)$.

$$\mathcal{F}\{\psi(x,y,t)\} = \mathcal{F}\left\{\sum_{n=1}^{\infty} A_n(x,t)Y_n(y)\right\}$$

The Fourier transform is a linear operator, so it can be brought inside the sum. Also, it only affects functions dependent on x.

$$\Psi(k,y,t) = \sum_{n=1}^{\infty} \mathcal{F}\left\{A_n(x,t)\right\} Y_n(y)$$

Replace $\mathcal{F}\{A\}$ with $\overline{A}$.

$$\Psi(k,y,t) = \sum_{n=1}^{\infty} \overline{A}_n(k,t)Y_n(y)$$

Substitute the eigenfunctions.

$$\Psi(k,y,t)=\sum_{n=1}^{\infty}B_n(k)e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}\sin n\pi y$$

Set $t=0$ and use equation (3).

$$\Psi(k,y,0)=\sum_{n=1}^{\infty}B_n(k)\sin n\pi y=\Psi_0(k,y)$$

We can determine $B_n(k)$ by taking advantage of the orthogonality of the sine function. Multiply both sides by $\sin m\pi y$, where m is a positive integer like n.

$$\sum_{n=1}^{\infty}B_n(k)\sin n\pi y\sin m\pi y=\Psi_0(k,y)\sin m\pi y$$

Now integrate both sides with respect to y over the domain it is defined for.

$$\int_0^1\sum_{n=1}^{\infty}B_n(k)\sin n\pi y\sin m\pi y\,dy=\int_0^1\Psi_0(k,y)\sin m\pi y\,dy$$

Bring the integral inside the sum.

$$\sum_{n=1}^{\infty}B_n(k)\underbrace{\int_0^1\sin n\pi y\sin m\pi y\,dy}_{=\frac{1}{2}\delta_{nm}}=\int_0^1\Psi_0(k,y)\sin m\pi y\,dy$$

The only term in the sum that isn't zero is the one where $n=m$, and the integral of sine squared is known to be 1/2.

$$\frac{1}{2}B_n(k)=\int_0^1\Psi_0(k,y)\sin n\pi y\,dy$$

Multiply both sides by 2 to solve for $B_n(k)$.

$$B_n(k)=2\int_0^1\Psi_0(k,y)\sin n\pi y\,dy$$

Plug $B_n(k)$ into the formula for $\Psi(k,y,t)$.

$$\Psi(k,y,t)=\sum_{n=1}^{\infty}\left[2\int_0^1\Psi_0(k,y')\sin n\pi y'\,dy'\right]e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}\sin n\pi y$$

Interchange the order of the sum and the integral.

$$\Psi(k,y,t)=\int_0^1\sum_{n=1}^{\infty}2\sin n\pi y'\sin n\pi y\,e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}\Psi_0(k,y')\,dy'$$

Let

$$\boxed{G(k,y,y',t)=\sum_{n=1}^{\infty}2\sin n\pi y'\sin n\pi y\,e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}.}$$

Then the formula can be expressed compactly like so.

$$\Psi(k,y,t) = \int_0^1 G(k,y,y',t)\Psi_0(k,y')\,dy'$$

$G(k,y,y',t)$ is known as the Green's function. With $\Psi(k,y,t)$ known, all we have to do now is take the inverse Fourier transform to find the general solution for $\psi(x,y,t)$.

$$\boxed{\psi(x,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^1 G(k,y,y',t)\Psi_0(k,y')e^{ikx}\,dy'\,dk,}$$

where

$$\boxed{\Psi_0(k,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\psi_0(x,y)\,dx.}$$

The Special Initial Condition

If

$$\psi_0(x,y) = \psi_{0n}(x) = \frac{1}{\alpha\sqrt{2}}e^{ik_0x-\frac{x^2}{a^2}},$$

then taking the Fourier transform of it yields

$$\Psi_0(k,y) = \frac{a}{2\alpha}e^{-\frac{1}{4}(k-k_0)^2}.$$

Plug this result and the Green's function into the formula for $\psi(x,y,t)$.

$$\psi(x,y,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^1 \sum_{n=1}^{\infty}\cancel{2}\sin n\pi y' \sin n\pi y\, e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}\frac{a}{\cancel{2}\alpha}e^{-\frac{1}{4}(k-k_0)^2}e^{ikx}\,dy'\,dk$$

Pull the constants out in front of the integrals.

$$\psi(x,y,t) = \frac{a}{\alpha\sqrt{2\pi}}\sum_{n=1}^{\infty}\sin n\pi y\int_{-\infty}^{\infty} e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}e^{-\frac{1}{4}(k-k_0)^2}e^{ikx}\int_0^1 \sin n\pi y'\,dy'\,dk$$

Evaluate the integral in dy'.

$$\psi(x,y,t) = \frac{a}{\alpha\sqrt{2\pi}}\sum_{n=1}^{\infty}\sin n\pi y\int_{-\infty}^{\infty} e^{\frac{ik\beta}{k^2+\kappa^2+(n\pi)^2}t}e^{-\frac{1}{4}(k-k_0)^2}e^{ikx}\left[\frac{1+(-1)^{n+1}}{n\pi}\right]dk$$

Bring the constant out in front and write the exponential functions like so.

$$\psi(x,y,t) = \frac{a}{\alpha\sqrt{2\pi}}\sum_{n=1}^{\infty}\left[\frac{1+(-1)^{n+1}}{n\pi}\right]\sin n\pi y\int_{-\infty}^{\infty} e^{-\frac{1}{4}(k-k_0)^2}e^{i[kx-\omega(k)t]}\,dk,$$

where $\omega(k)$ is the dispersion relation.

$$\omega(k) = -\frac{k\beta}{k^2+\kappa^2+(n\pi)^2}$$

The integral is too complicated to be evaluated explicitly, but the method of stationary phase can be used to determine the leading order behavior of $\psi(x,y,t)$ as $t\to\infty$.