Exercise 3

For what values of m is $u_{xx} - m_x u_{xy} + 4x^2 u_{yy} = 0$ (a) hyperbolic, (b) parabolic, or (c) elliptic? For $m = 0$, reduce to canonical form.

Solution

$u_{xx} - m_x u_{xy} + 4x^2 u_{yy} = 0$

Comparing this equation with the general form of a second-order PDE,
$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, we see that $A = 1$, $B = -m_x$, $C = 4x^2$, $D = 0$, $E = 0$, $F = 0$, and $G = 0$. Note that the discriminant, $B^2 - 4AC = m_x^2 - 16x^2$, can be positive, zero, or negative, depending on whether $m_x^2 - 16x^2 > 0$, $m_x^2 - 16x^2 = 0$, or $m_x^2 - 16x^2 < 0$, respectively. That is,

$$\text{The PDE is} \begin{cases} \text{hyperbolic} & \text{if } m_x^2 - 16x^2 > 0. \\ \text{parabolic} & \text{if } m_x^2 - 16x^2 = 0. \\ \text{elliptic} & \text{if } m_x^2 - 16x^2 < 0. \end{cases}$$

Let us consider each case individually.

Case I: The PDE is hyperbolic $(m_x^2 - 16x^2 > 0)$

$$m_x^2 - 16x^2 > 0$$
$$m_x^2 > 16x^2$$

Taking the square root of both sides gives

$$|m_x| > 4|x|.$$

Breaking this into two inequalities gets rid of the absolute value sign on m_x:

$$m_x > 4|x| \quad \textbf{or} \quad m_x < -4|x|.$$

To get rid of the absolute value signs on x, we have to consider the cases where $x < 0$ and $x > 0$. For $x < 0$,

$$m_x > -4x \quad \textbf{or} \quad m_x < 4x.$$

Integrating these two inequalities partially with respect to x gives

$$m(x, y) > -2x^2 + f_1(y) \quad \textbf{or} \quad m(x, y) < 2x^2 + f_2(y),$$

where f_1 and f_2 are arbitrary differentiable functions of y; that is, they are of class C^1. Let A be the set of all functions $m(x, y)$ that satisfy $m(x, y) > -2x^2 + f_1(y)$, and let B be the set of all functions $m(x, y)$ that satisfy $m(x, y) < 2x^2 + f_2(y)$.

$$A = \left\{m(x, y) \mid m(x, y) > -2x^2 + f_1(y),\ x < 0,\ y \in \mathbb{R},\ f_1 \in C^1\right\}$$
$$B = \left\{m(x, y) \mid m(x, y) < 2x^2 + f_2(y),\ x < 0,\ y \in \mathbb{R},\ f_2 \in C^1\right\}$$

For $x > 0$,

$$m_x > 4x \quad \textbf{or} \quad m_x < -4x.$$

Integrating these two inequalities partially with respect to x gives

$$m(x,y) > 2x^2 + f_3(y) \quad \textbf{or} \quad m(x,y) < -2x^2 + f_4(y),$$

where f_3 and f_4 are arbitrary differentiable functions of y; that is, they are of class C^1. Let C be the set of all functions $m(x,y)$ that satisfy $m(x,y) > 2x^2 + f_3(y)$, and let D be the set of all functions $m(x,y)$ that satisfy $m(x,y) < -2x^2 + f_4(y)$.

$$C = \left\{m(x,y) \mid m(x,y) > 2x^2 + f_3(y),\ x > 0,\ y \in \mathbb{R},\ f_3 \in C^1\right\}$$
$$D = \left\{m(x,y) \mid m(x,y) < -2x^2 + f_4(y),\ x > 0,\ y \in \mathbb{R},\ f_4 \in C^1\right\}$$

Therefore, the PDE is hyperbolic for the following set of values of $m(x,y)$:

$$\{m(x,y) \mid m(x,y) \in (A \cup B) \cup (C \cup D)\}.$$

Case II: The PDE is parabolic $(m_x^2 - 16x^2 = 0)$

$$m_x^2 - 16x^2 = 0$$
$$m_x^2 = 16x^2$$

Taking the square root of both sides gives

$$|m_x| = 4|x|.$$

Breaking this into two equations gets rid of the absolute value sign on m_x:

$$m_x = 4|x| \quad \textbf{or} \quad m_x = -4|x|.$$

To get rid of the absolute value signs on x, we have to consider the cases where $x < 0$ and $x > 0$. For $x < 0$,

$$m_x = -4x \quad \textbf{or} \quad m_x = 4x.$$

Integrating these two equations partially with respect to x gives

$$m(x,y) = -2x^2 + f_5(y) \quad \textbf{or} \quad m(x,y) = 2x^2 + f_6(y),$$

where f_5 and f_6 are arbitrary differentiable functions of y; that is, they are of class C^1. Let E be the set of all functions $m(x,y)$ that satisfy $m(x,y) = -2x^2 + f_5(y)$, and let F be the set of all functions $m(x,y)$ that satisfy $m(x,y) = 2x^2 + f_6(y)$.

$$E = \left\{m(x,y) \mid m(x,y) = -2x^2 + f_5(y),\ x < 0,\ y \in \mathbb{R},\ f_5 \in C^1\right\}$$
$$F = \left\{m(x,y) \mid m(x,y) = 2x^2 + f_6(y),\ x < 0,\ y \in \mathbb{R},\ f_6 \in C^1\right\}$$

For $x > 0$,

$$m_x = 4x \quad \textbf{or} \quad m_x = -4x.$$

Integrating these two equations partially with respect to x gives

$$m(x,y) = 2x^2 + f_7(y) \quad \textbf{or} \quad m(x,y) = -2x^2 + f_8(y),$$

where f_7 and f_8 are arbitrary differentiable functions of y; that is, they are of class C^1. Let G be the set of all functions $m(x,y)$ that satisfy $m(x,y) = 2x^2 + f_7(y)$, and let H be the set of all functions $m(x,y)$ that satisfy $m(x,y) = -2x^2 + f_8(y)$.

$$G = \left\{m(x,y) \mid m(x,y) = 2x^2 + f_7(y),\ x > 0,\ y \in \mathbb{R},\ f_7 \in C^1\right\}$$
$$H = \left\{m(x,y) \mid m(x,y) = -2x^2 + f_8(y),\ x > 0,\ y \in \mathbb{R},\ f_8 \in C^1\right\}$$

Therefore, the PDE is parabolic for the following set of values of $m(x,y)$:

$$\{m(x,y) \mid m(x,y) \in (E \cup F) \cup (G \cup H)\}.$$

Case III: The PDE is elliptic $(m_x^2 - 16x^2 < 0)$

$$m_x^2 - 16x^2 < 0$$
$$m_x^2 < 16x^2$$

Taking the square root of both sides gives

$$|m_x| < 4|x|.$$

Breaking this into two inequalities gets rid of the absolute value sign on m_x:

$$-4|x| < m_x < 4|x|$$

$$m_x < 4|x| \quad \textbf{and} \quad m_x > -4|x|.$$

To get rid of the absolute value signs on x, we have to consider the cases where $x < 0$ and $x > 0$. For $x < 0$,

$$m_x < -4x \quad \textbf{and} \quad m_x > 4x.$$

Integrating these two inequalities partially with respect to x gives

$$m(x,y) < -2x^2 + f_9(y) \quad \textbf{and} \quad m(x,y) > 2x^2 + f_{10}(y),$$

where f_9 and f_{10} are arbitrary differentiable functions of y; that is, they are of class C^1. Let I be the set of all functions $m(x,y)$ that satisfy $m(x,y) < -2x^2 + f_9(y)$, and let J be the set of all functions $m(x,y)$ that satisfy $m(x,y) > 2x^2 + f_{10}(y)$.

$$I = \left\{m(x,y) \mid m(x,y) < -2x^2 + f_9(y),\ x < 0,\ y \in \mathbb{R},\ f_9 \in C^1\right\}$$
$$J = \left\{m(x,y) \mid m(x,y) > 2x^2 + f_{10}(y),\ x < 0,\ y \in \mathbb{R},\ f_{10} \in C^1\right\}$$

For $x > 0$,

$$m_x < 4x \quad \textbf{and} \quad m_x > -4x.$$

Integrating these two inequalities partially with respect to x gives

$$m(x,y) < 2x^2 + f_{11}(y) \quad \textbf{and} \quad m(x,y) > -2x^2 + f_{12}(y).$$

where f_{11} and f_{12} are arbitrary differentiable functions of y; that is, they are of class C^1. Let K be the set of all functions $m(x,y)$ that satisfy $m(x,y) < 2x^2 + f_{11}(y)$, and let L be the set of all functions $m(x,y)$ that satisfy $m(x,y) > -2x^2 + f_{12}(y)$.

$$K = \left\{m(x,y) \mid m(x,y) < 2x^2 + f_{11}(y),\ x > 0,\ y \in \mathbb{R},\ f_{11} \in C^1\right\}$$
$$L = \left\{m(x,y) \mid m(x,y) > -2x^2 + f_{12}(y),\ x > 0,\ y \in \mathbb{R},\ f_{12} \in C^1\right\}$$

Therefore, the PDE is elliptic for the following set of values of $m(x,y)$:

$$\{m(x,y) \mid m(x,y) \in (I \cap J) \cup (K \cap L)\}.$$

Case IV: $m = 0$

If $m = 0$, then $B^2 - 4AC = -16x^2$ for all x, and the PDE is **elliptic**. The two distinct families of characteristic curves, therefore, lie in the complex plane. The characteristic equations are given by

$$\frac{dy}{dx} = \frac{1}{2A}\left(B \pm \sqrt{B^2 - 4AC}\right)$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\pm\sqrt{-16x^2}\right)$$
$$\frac{dy}{dx} = \pm 2ix.$$

Integrating the characteristic equations, we find that

$$y(x) = \pm ix^2 + C_0.$$

Solving for the constant of integration (or any convenient multiple thereof),

$$\text{Working with } -ix^2\text{:}\quad C_0 = y + ix^2 = \phi(x,y)$$
$$\text{Working with } +ix^2\text{:}\quad C_0 = y - ix^2 = \psi(x,y).$$

The PDE does not reduce to the canonical form for an elliptic equation with the typical change of variables, $\xi = \phi(x,y) = y + ix^2$ and $\eta = \psi(x,y) = y - ix^2$. Since ξ and η are complex conjugates of each other, we introduce the new real variables,

$$\alpha = \frac{1}{2}(\xi + \eta) = y$$
$$\beta = \frac{1}{2i}(\xi - \eta) = x^2,$$

which transform the PDE to the canonical form. After changing variables $(x,y) \to (\alpha,\beta)$, the PDE becomes

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + D^{**}u_{\alpha} + E^{**}u_{\beta} + F^{**}u = G^{**},$$

where, using the chain rule,

$$
\begin{aligned}
A^{**} &= A\alpha_x^2 + B\alpha_x\alpha_y + C\alpha_y^2 \\
B^{**} &= 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y \\
C^{**} &= A\beta_x^2 + B\beta_x\beta_y + C\beta_y^2 \\
D^{**} &= A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y \\
E^{**} &= A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y \\
F^{**} &= F \\
G^{**} &= G.
\end{aligned}
$$

Plugging in the numbers and derivatives to these equations, we find that $A^{**} = 4x^2 = 4\beta$, $B^{**} = 0$, $C^{**} = 4x^2 = 4\beta$, $D^{**} = 0$, $E^{**} = 2$, $F^{**} = 0$, and $G^{**} = 0$. Thus, the PDE simplifies to

$$
\begin{aligned}
4\beta u_{\alpha\alpha} + 4\beta u_{\beta\beta} + 2u_\beta = 0 \\
u_{\alpha\alpha} + u_{\beta\beta} = -\frac{1}{2\beta}u_\beta
\end{aligned}
$$

This is the canonical form of the PDE when $m = 0$.