Exercise 36

Obtain the solution of the Stokes-Ekman problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid bounded by an infinite horizontal disk at $z = 0$ when both the fluid and the disk rotate with a uniform angular velocity Ω about the z-axis. The governing boundary layer equation and the boundary and the initial conditions are

$$\begin{aligned} \frac{\partial q}{\partial t} + 2\Omega i q = \nu \frac{\partial^2 q}{\partial z^2}, \quad z > 0,\ t > 0, \\ q(z,t) = ae^{i\omega t} + be^{-i\omega t} \quad \text{on } z = 0,\ t > 0, \\ q(z,t) \to 0 \quad \text{as } z \to \infty,\ t > 0, \\ q(z,t) = 0 \quad \text{at } t \le 0,\ \text{for all } z > 0, \end{aligned}$$

where $q = u + iv$, ω is the frequency of oscillations of the disk, and a, b are complex constants. Hence, deduce the steady-state solution and determine the structure of the associated boundary layers.

Solution

The PDE is defined for $t > 0$ and we have an initial condition, so the Laplace transform can be used to solve it. It is defined as

$$\mathcal{L}\{q(z,t)\} = \bar{q}(z,s) = \int_0^t e^{-st} q(z,t)\, dt,$$

which means the derivatives of q with respect to z and t transform as follows.

$$\begin{aligned} \mathcal{L}\left\{\frac{\partial^n q}{\partial z^n}\right\} &= \frac{d^n \bar{q}}{dz^n} \\ \mathcal{L}\left\{\frac{\partial q}{\partial t}\right\} &= s\bar{q}(z,s) - q(z,0) \end{aligned}$$

Take the Laplace transform of both sides of the PDE.

$$\mathcal{L}\{q_t + 2\Omega i q\} = \mathcal{L}\{\nu q_{zz}\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\{q_t\} + 2\Omega i \mathcal{L}\{q\} = \nu \mathcal{L}\{q_{zz}\}$$

Transform the derivatives with the relations above.

$$s\bar{q}(z,s) - q(z,0) + 2\Omega i \bar{q}(z,s) = \nu \frac{d^2 \bar{q}}{dz^2}$$

From the initial condition, $q(z,t) = 0$ for $t \le 0$, we have $q(z,0) = 0$.

$$\frac{d^2 \bar{q}}{dz^2} = \frac{s + 2\Omega i}{\nu} \bar{q}$$

The PDE has thus been reduced to an ODE whose solution can be written in terms of exponential functions.

$$\bar{q}(z,s) = A(s) e^{\sqrt{\frac{s+2\Omega i}{\nu}} z} + B(s) e^{-\sqrt{\frac{s+2\Omega i}{\nu}} z}$$

In order to satisfy the condition that $q(z,t) \to 0$ as $z \to \infty$, we require that $A(s) = 0$.

$$\bar{q}(z,s) = B(s)e^{-\sqrt{\frac{s+2\Omega i}{\nu}}z}$$

To determine $B(s)$ we have to use the boundary condition at $z = 0$, $q(0,t) = ae^{i\omega t} + be^{-i\omega t}$. Take the Laplace transform of both sides of it.

$$\mathcal{L}\{q(0,t)\} = \mathcal{L}\{ae^{i\omega t} + be^{-i\omega t}\}$$

$$\bar{q}(0,s) = \frac{a}{s - i\omega} + \frac{b}{s + i\omega} \tag{1}$$

Setting $z = 0$ in the formula for $\bar{q}$ and using equation (1), we have

$$\bar{q}(0,s) = B(s) = \frac{a}{s - i\omega} + \frac{b}{s + i\omega}.$$

Thus,

$$\bar{q}(z,s) = \left(\frac{a}{s - i\omega} + \frac{b}{s + i\omega}\right) e^{-\sqrt{\frac{s+2\Omega i}{\nu}}z}.$$

Now that we have $\bar{q}(z,s)$, we can get $q(z,t)$ by taking the inverse Laplace transform of it.

$$q(z,t) = \mathcal{L}^{-1}\{\bar{q}(z,s)\}$$

The convolution theorem can be used to write an integral solution for $q(z,t)$. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

The inverse Laplace transform of the individual functions are

$$\mathcal{L}^{-1}\left\{\frac{a}{s - i\omega} + \frac{b}{s + i\omega}\right\} = ae^{i\omega t} + be^{-i\omega t}$$

$$\mathcal{L}^{-1}\left\{e^{-\sqrt{\frac{s+2\Omega i}{\nu}}z}\right\} = \frac{z}{\sqrt{4\pi\nu t^3}}e^{-\frac{z^2}{4\nu t} - 2\Omega i t},$$

so by the convolution theorem, we have for $q(z,t)$

$$q(z,t) = \int_0^t [ae^{i\omega(t-\tau)} + be^{-i\omega(t-\tau)}]\frac{z}{\sqrt{4\pi\nu\tau^3}}e^{-\frac{z^2}{4\nu\tau} - 2\Omega i\tau}\,d\tau.$$

Rewrite the integral as follows.

$$q(z,t) = \frac{z}{\sqrt{4\pi\nu}}\left[ae^{i\omega t}\int_0^t \frac{1}{\tau^{3/2}}e^{-\frac{z^2}{4\nu\tau} - i(2\Omega+\omega)\tau}\,d\tau + be^{-i\omega t}\int_0^t \frac{1}{\tau^{3/2}}e^{-\frac{z^2}{4\nu\tau} - i(2\Omega-\omega)\tau}\,d\tau\right]$$

Evaluating the integrals and simplifying, we get

$$q(z,t) = \frac{ae^{i\omega t}}{2}\left[e^{-\sqrt{\frac{i(2\Omega+\omega)}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} - \sqrt{i(2\Omega+\omega)t}\right) + e^{\sqrt{\frac{i(2\Omega+\omega)}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} + \sqrt{i(2\Omega+\omega)t}\right)\right]$$
$$+ \frac{be^{-i\omega t}}{2}\left[e^{-\sqrt{\frac{i(2\Omega-\omega)}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} - \sqrt{i(2\Omega-\omega)t}\right) + e^{\sqrt{\frac{i(2\Omega-\omega)}{\nu}}z}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} + \sqrt{i(2\Omega-\omega)t}\right)\right],$$

where erfc is the complementary error function, a known special function, defined as

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-r^2}\, dr.$$

In order to satisfy the condition that $q(z,t) = 0$ for $t \leq 0$, we write the solution as a piecewise function.

$$q(z,t) = \begin{cases} 0 & t \leq 0 \\ \text{The gigantic expression for } q & t > 0 \end{cases}$$

This can be written compactly with the Heaviside function. Therefore,

$$q(z,t) = \left\{ \frac{ae^{i\omega t}}{2} \left[e^{-\sqrt{\frac{i(2\Omega+\omega)}{\nu}}z} \operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} - \sqrt{i(2\Omega+\omega)t} \right) + e^{\sqrt{\frac{i(2\Omega+\omega)}{\nu}}z} \operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} + \sqrt{i(2\Omega+\omega)t} \right) \right] \right.$$
$$\left. + \frac{be^{-i\omega t}}{2} \left[e^{-\sqrt{\frac{i(2\Omega-\omega)}{\nu}}z} \operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} - \sqrt{i(2\Omega-\omega)t} \right) + e^{\sqrt{\frac{i(2\Omega-\omega)}{\nu}}z} \operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}} + \sqrt{i(2\Omega-\omega)t} \right) \right] \right\} H(t)$$

The solution at the back of the book does not include $H(t)$ and hence is only valid for $t > 0$. Also, there is a typo; in the last erfc function under the second square root it says $2\Omega + \omega$, but this is incorrect. It should be $2\Omega - \omega$ as written here.