Exercise 36

Obtain the solution of the Stokes-Ekman problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid bounded by an infinite horizontal disk at $z = 0$ when both the fluid and the disk rotate with a uniform angular velocity Ω about the z-axis. The governing boundary layer equation and the boundary and the initial conditions are

$$\begin{aligned}
&\frac{\partial q}{\partial t} + 2\Omega i q = \nu \frac{\partial^2 q}{\partial z^2}, \quad z > 0,\ t > 0, \\
&\qquad q(z,t) = ae^{i\omega t} + be^{-i\omega t} \quad \text{on } z = 0,\ t > 0, \\
&\qquad q(z,t) \to 0 \quad \text{as } z \to \infty,\ t > 0, \\
&\qquad q(z,t) = 0 \quad \text{at } t \leq 0, \text{ for all } z > 0,
\end{aligned}$$

where $q = u + iv$, ω is the frequency of oscillations of the disk, and a, b are complex constants. Hence, deduce the steady-state solution and determine the structure of the associated boundary layers.