# Exercise 42

Show that the solution of the boundary-value problem

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 < r < \infty,\ 0 < z < \infty,$$
$$u(r,z) = \frac{1}{\sqrt{a^2 + r^2}} \quad \text{on } z = 0,\ 0 < r < \infty,$$

is

$$u(r,z) = \int_0^\infty e^{-\kappa(z+a)} J_0(\kappa r)\, d\kappa = \frac{1}{\sqrt{(z+a)^2 + r^2}}.$$

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**Solution**

The PDE is defined for $0 < r < \infty$, so the Hankel transform can be applied to solve it. The zero-order Hankel transform is defined as

$$\mathcal{H}_0\{u(r,z)\} = \tilde{u}(\kappa, z) = \int_0^\infty r J_0(\kappa r) u(r,z)\, dr,$$

where $J_0(\kappa r)$ is the Bessel function of order 0. Hence, the radial part of the laplacian in cylindrical coordinates transforms as follows.

$$\mathcal{H}_0\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\} = -\kappa^2 \tilde{u}(\kappa, z)$$

The partial derivative with respect to $z$ transforms like so.

$$\mathcal{H}_0\left\{\frac{\partial^n u}{\partial z^n}\right\} = \frac{d^n \tilde{u}}{dz^n}$$

Take the zero-order Hankel transform of both sides of the PDE.

$$\mathcal{H}_0\left\{u_{rr} + \frac{1}{r}u_r + u_{zz}\right\} = \mathcal{H}_0\{0\}$$

The Hankel transform is a linear operator.

$$\mathcal{H}_0\left\{u_{rr} + \frac{1}{r}u_r\right\} + \mathcal{H}_0\{u_{zz}\} = 0$$

Use the relations above to transform the derivatives.

$$-\kappa^2 \tilde{u} + \frac{d^2 \tilde{u}}{dz^2} = 0$$

Bring the term with $\tilde{u}$ to the other side.

$$\frac{d^2 \tilde{u}}{dz^2} = \kappa^2 \tilde{u}$$

The PDE has thus been reduced to an ODE whose solution can be written in terms of exponential functions.

$$\tilde{u}(\kappa, z) = A(\kappa) e^{|\kappa| z} + B(\kappa) e^{-|\kappa| z}$$

In order to keep $\tilde{u}$ bounded as $z \to \infty$, we require that $A(\kappa) = 0$.

$$\tilde{u}(\kappa, z) = B(\kappa)e^{-|\kappa|z} \tag{1}$$

Use the provided boundary condition at $z = 0$ to determine $B(\kappa)$.

$$u(r,0) = \frac{1}{\sqrt{a^2 + r^2}} \quad \rightarrow \quad \mathcal{H}_0\{u(r,0)\} = \mathcal{H}_0\left\{\frac{1}{\sqrt{a^2+r^2}}\right\}$$

$$\tilde{u}(\kappa, 0) = \frac{e^{-a\kappa}}{\kappa} \tag{2}$$

Plugging in $z = 0$ into equation (1) and using equation (2), we get

$$\tilde{u}(\kappa, 0) = B(\kappa) = \frac{e^{-a\kappa}}{\kappa}.$$

Thus,

$$\tilde{u}(\kappa, z) = \frac{e^{-a\kappa}}{\kappa}e^{-|\kappa|z}.$$

Now that we have $\tilde{u}(\kappa, z)$, we can get $u(r, z)$ by taking the inverse Hankel transform of it.

$$u(r, z) = \mathcal{H}_0^{-1}\{\tilde{u}(\kappa, z)\}$$

It is defined as

$$\mathcal{H}_0^{-1}\{\tilde{u}(\kappa, z)\} = \int_0^\infty \kappa J_0(\kappa r)\tilde{u}(\kappa, z)\, d\kappa,$$

so we have

$$u(r, z) = \int_0^\infty \kappa J_0(\kappa r)\frac{e^{-(z+a)\kappa}}{\kappa}\, d\kappa.$$

The absolute value sign on $\kappa$ has been dropped because it is positive. Cancel $\kappa$.

$$u(r, z) = \int_0^\infty e^{-(z+a)\kappa} J_0(\kappa r)\, d\kappa$$

Use the known integral

$$\int_0^\infty e^{-a\kappa} J_0(\kappa r)\, d\kappa = \frac{1}{\sqrt{a^2 + r^2}}.$$

Therefore,

$$u(r, z) = \frac{1}{\sqrt{(z + a)^2 + r^2}}.$$