Exercise 44

If $f(r) = A(a^2 + r^2)^{-\frac{1}{2}}$, where A is a constant, show that the solution of the biharmonic equation described in Example 1.10.7 is

$$u(r, z) = A\frac{\{r^2 + (z + a)(2z + a)\}}{[r^2 + (z + a)^2]^{3/2}}.$$

Solution

The PDE we have to solve is the axisymmetric biharmonic equation,

$$\nabla^4 u(r, z) = 0, \quad 0 \le r < \infty, \ z > 0,$$

subject to the boundary conditions,

$$\begin{aligned} u(r, 0) = f(r) &= \frac{A}{\sqrt{a^2 + r^2}}, \quad 0 \le r < \infty, \\ \frac{\partial u}{\partial z} &= 0 \quad \text{on } z = 0, \ 0 \le r < \infty, \\ u(r, z) &\to \infty \quad \text{as } r \to \infty. \end{aligned}$$

Since $0 \le r < \infty$, the Hankel transform can be applied to solve it. The zero-order Hankel transform is defined as

$$\mathcal{H}_0\{u(r, z)\} = \tilde{u}(\kappa, z) = \int_0^\infty r J_0(\kappa r) u(r, z)\, dr,$$

where $J_0(\kappa r)$ is the Bessel function of order 0. Hence, the radial part of the laplacian in cylindrical coordinates transforms as follows.

$$\mathcal{H}_0\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\} = -\kappa^2 \tilde{u}(\kappa, z)$$

The partial derivative with respect to z transforms like so.

$$\mathcal{H}_0\left\{\frac{\partial^n u}{\partial z^n}\right\} = \frac{d^n \tilde{u}}{dz^n}$$

∇^4 is the laplacian operator squared. In cylindrical coordinates, the PDE takes the form

$$\nabla^4 u = (\nabla^2)^2 u = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 u = 0.$$

Take the zero-order Hankel transform of both sides of the PDE.

$$\mathcal{H}_0\left\{\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 u\right\} = \mathcal{H}_0\{0\}$$

Use the relations above to transform the partial derivatives.

$$\left(-\kappa^2 + \frac{d^2}{dz^2}\right)^2 \tilde{u}(\kappa, z) = 0$$

Expand the operator acting on $\tilde{u}$.

$$\left(\frac{d^4}{dz^4} - 2\kappa^2\frac{d^2}{dz^2} + \kappa^4\right)\tilde{u} = 0$$

Distribute the operator.

$$\frac{d^4\tilde{u}}{dz^4} - 2\kappa^2\frac{d^2\tilde{u}}{dz^2} + \kappa^4\tilde{u} = 0 \tag{1}$$

The PDE has thus been reduced to a fourth-order homogeneous ODE with constant coefficients. The standard procedure for solving it is to assume a solution of the form, $\tilde{u} = e^{pz}$, and then substitute it into the ODE to determine p.

$$\tilde{u} = e^{pz} \quad \rightarrow \quad \frac{d\tilde{u}}{dz} = pe^{pz} \quad \rightarrow \quad \frac{d^2\tilde{u}}{dz^2} = p^2e^{pz} \quad \rightarrow \quad \frac{d^3\tilde{u}}{dz^3} = p^3e^{pz} \quad \rightarrow \quad \frac{d^4\tilde{u}}{dz^4} = p^4e^{pz}$$

Substituting these expressions into the ODE, we get

$$p^4e^{pz} - 2\kappa^2p^2e^{pz} + \kappa^4e^{pz} = 0.$$

Divide both sides by e^{pz} to get an algebraic equation for p.

$$p^4 - 2\kappa^2p^2 + \kappa^4 = 0$$

Factor the left side.

$$(p+\kappa)^2(p-\kappa)^2 = 0$$

Hence,

$$p = -\kappa \text{ (multiplicity 2)} \quad p = \kappa \text{ (multiplicity 2)},$$

which means the solution to the ODE in equation (1) is

$$\tilde{u}(\kappa, z) = C_1(\kappa)e^{-\kappa z} + C_2(\kappa)ze^{-\kappa z} + C_3(\kappa)e^{\kappa z} + C_4(\kappa)ze^{\kappa z}. \tag{2}$$

Since $\tilde{u}(\kappa, z)$ must remain bounded as $z \to \infty$, we require $C_3(\kappa) = 0$ and $C_4(\kappa) = 0$. To determine $C_1(\kappa)$ and $C_2(\kappa)$, make use of the provided boundary conditions at $z = 0$. Take the zero-order Hankel transform of both sides of them.

$$u(r, 0) = \frac{A}{\sqrt{a^2 + r^2}} \quad \rightarrow \quad \mathcal{H}_0\{u(r, 0)\} = \mathcal{H}_0\left\{\frac{A}{\sqrt{a^2+r^2}}\right\}$$

$$\tilde{u}(\kappa, 0) = \frac{A}{\kappa}e^{-\kappa a} \tag{3}$$

$$\frac{\partial u}{\partial z}(r, 0) = 0 \quad \rightarrow \quad \mathcal{H}_0\left\{\frac{\partial u}{\partial z}\right\} = \mathcal{H}_0\{0\}$$

$$\frac{d\tilde{u}}{dz}(\kappa, 0) = 0 \tag{4}$$

Setting $z = 0$ in equation (2) and using equation (3), we get

$$\tilde{u}(\kappa, 0) = C_1(\kappa) = \frac{A}{\kappa}e^{-\kappa a}.$$

Taking the derivative of $\tilde{u}(\kappa, z)$ with respect to z, setting $z = 0$, and using equation (4), we get

$$\frac{d\tilde{u}}{dz}(\kappa, 0) = C_2(\kappa) - Ae^{-\kappa a} = 0 \quad \rightarrow \quad C_2(\kappa) = Ae^{-\kappa a}.$$

With the constants determined, we now know $\tilde{u}$.

$$\begin{aligned}\tilde{u}(\kappa, z) &= \frac{A}{\kappa}e^{-\kappa a}e^{-\kappa z} + Ae^{-\kappa a}ze^{-\kappa z} \\ &= \frac{A}{\kappa}(1+\kappa z)e^{-\kappa(z+a)}\end{aligned}$$

All that's left to do is to take the inverse Hankel transform of this to get $u(r, z)$.

$$u(r, z) = \mathcal{H}_0^{-1}\{\tilde{u}(\kappa, z)\}$$

It is defined as

$$\mathcal{H}_0^{-1}\{\tilde{u}(\kappa, z)\} = \int_0^\infty \kappa J_0(\kappa r)\tilde{u}(\kappa, z)\, d\kappa,$$

so

$$u(r, z) = \int_0^\infty \kappa J_0(\kappa r)\frac{A}{\kappa}(1+\kappa z)e^{-\kappa(z+a)}\, d\kappa.$$

Cancel κ and shuffle the terms in the integrand.

$$u(r, z) = \int_0^\infty A(1+\kappa z)e^{-\kappa(z+a)}J_0(\kappa r)\, d\kappa$$

Split up the integral into two and bring the constants out in front of them.

$$u(r, z) = A\int_0^\infty e^{-\kappa(z+a)}J_0(\kappa r)\, d\kappa + zA\int_0^\infty \kappa e^{-\kappa(z+a)}J_0(\kappa r)\, d\kappa$$

We can evaluate both these integrals from the known integral,

$$\int_0^\infty e^{-\kappa a}J_0(\kappa r)\, d\kappa = \frac{1}{\sqrt{r^2+a^2}}.$$

Differentiate both sides with respect to a.

$$\int_0^\infty (-\kappa)e^{-\kappa a}J_0(\kappa r)\, d\kappa = -\frac{a}{(r^2+a^2)^{3/2}} \quad \rightarrow \quad \int_0^\infty \kappa e^{-\kappa a}J_0(\kappa r)\, d\kappa = \frac{a}{(r^2+a^2)^{3/2}}$$

With these two integrals, we can obtain $u(r, z)$.

$$u(r, z) = A\frac{1}{\sqrt{r^2+(z+a)^2}} + zA\frac{z+a}{[r^2+(z+a)^2]^{3/2}}$$

Multiply the numerator and denominator of the first fraction by $r^2+(z+a)^2$ to get a common denominator.

$$u(r, z) = \frac{A[r^2+(z+a)^2] + zA(z+a)}{[r^2+(z+a)^2]^{3/2}}$$

Factor A and then factor $z+a$ from the last two terms in the numerator to get the final result.

$$u(r, z) = A\frac{r^2+(z+a)(2z+a)}{[r^2+(z+a)^2]^{3/2}}$$