Exercise 45

Solve the axisymmetric biharmonic equation for the free vibration of an elastic disk

$$b^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2 u + u_{tt} = 0, \quad 0 < r < \infty,\ t > 0,$$

$$u(r,0) = f(r), \quad u_t(r,0) = 0 \quad \text{for } 0 < r < \infty,$$

where $b^2 = \frac{D}{2\sigma h}$ is the ratio of the flexural rigidity of the disk and its mass $2h\sigma$ per unit area.

Solution

Since $0 < r < \infty$, the Hankel transform can be applied to solve it. The zero-order Hankel transform is defined as

$$\mathcal{H}_0\{u(r,t)\} = \tilde{u}(\kappa,t) = \int_0^\infty r J_0(\kappa r) u(r,t)\, dr,$$

where $J_0(\kappa r)$ is the Bessel function of order 0. Hence, the radial part of the laplacian in cylindrical coordinates transforms as follows.

$$\mathcal{H}_0\left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right\} = -\kappa^2\tilde{u}(\kappa,z)$$

The partial derivative with respect to t transforms like so.

$$\mathcal{H}_0\left\{\frac{\partial^n u}{\partial t^n}\right\} = \frac{d^n\tilde{u}}{dt^n}$$

Take the zero-order Hankel transform of both sides of the PDE.

$$\mathcal{H}_0\left\{b^2\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2 u + u_{tt}\right\} = \mathcal{H}_0\{0\}$$

The Hankel transform is a linear operator.

$$b^2\mathcal{H}_0\left\{\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2 u\right\} + \mathcal{H}_0\{u_{tt}\} = 0$$

Use the relations above to transform the partial derivatives.

$$b^2(-\kappa^2)^2\tilde{u}(\kappa,t) + \frac{d^2\tilde{u}}{dt^2} = 0$$

Move the term with $\tilde{u}$ to the other side.

$$\frac{d^2\tilde{u}}{dt^2} = -b^2\kappa^4\tilde{u}$$

The PDE has thus been reduced to an ODE whose solution can be expressed in terms of sine and cosine.

$$\tilde{u}(\kappa,t) = A(\kappa)\cos b\kappa^2 t + B(\kappa)\sin b\kappa^2 t \tag{1}$$

To determine the constants, $A(\kappa)$ and $B(\kappa)$, we have to use the provided initial conditions. Take the zero-order Hankel transform of both sides of them.

$$u(r,0) = f(r) \quad \to \quad \mathcal{H}_0\{u(r,0)\} = \mathcal{H}_0\{f(r)\}$$

$$\tilde{u}(\kappa,0) = \tilde{f}(\kappa) \tag{2}$$

$$\frac{\partial u}{\partial t}(r,0) = 0 \quad \to \quad \mathcal{H}_0\left\{\frac{\partial u}{\partial t}(r,0)\right\} = \mathcal{H}_0\{0\}$$

$$\frac{d\tilde{u}}{dt}(\kappa,0) = 0 \tag{3}$$

Plugging in $t = 0$ into equation (1) and using equation (2), we have

$$\tilde{u}(\kappa,0) = A(\kappa) = \tilde{f}(\kappa).$$

Taking the derivative of $\tilde{u}(\kappa,t)$ with respect to t, setting $t = 0$, and then using equation (3) gives us

$$\frac{d\tilde{u}}{dt}(\kappa,0) = b\kappa^2 B(\kappa) = 0 \quad \to \quad B(\kappa) = 0.$$

Thus,

$$\tilde{u}(\kappa,t) = \tilde{f}(\kappa)\cos b\kappa^2 t.$$

Now that we have $\tilde{u}(\kappa,t)$, we can change back to $u(r,t)$ by taking the inverse Hankel transform of it.

$$u(r,t) = \mathcal{H}_0^{-1}\{\tilde{u}(\kappa,t)\}$$

It is defined as

$$\mathcal{H}_0^{-1}\{\tilde{u}(\kappa,t)\} = \int_0^\infty \kappa J_0(\kappa r)\tilde{u}(\kappa,t)\,d\kappa.$$

Therefore,

$$u(r,t) = \int_0^\infty \kappa J_0(\kappa r)\tilde{f}(\kappa)\cos b\kappa^2 t\,d\kappa,$$

where

$$\tilde{f}(\kappa) = \int_0^\infty r J_0(\kappa r) f(r)\,dr.$$