Exercise 48

Solve the problem of the electrified unit disk in the (x, t)-plane with center at the origin. The electric potential $u(r, z)$ is axisymmetric and satisfies the boundary-value problem

$$u_{rr} + \frac{1}{r}u_r + u_{zz} = 0, \quad 0 < r < \infty,\ 0 < z < \infty,$$
$$u(r, 0) = u_0, \quad 0 \le r \le a,$$
$$\frac{\partial u}{\partial z} = 0, \quad \text{on } z = 0 \text{ for } a < r < \infty,$$
$$u(r, z) \to 0 \quad \text{as } z \to \infty \text{ for all } r,$$

where u_0 is constant. Show that the solution is

$$u(r, z) = \frac{2u_0}{\pi}\int_0^\infty J_0(kr)\frac{\sin ak}{k}e^{-kz}\,dk.$$

Solution

Since $0 < r < \infty$, the Hankel transform can be applied to solve it. The zero-order Hankel transform is defined as

$$\mathcal{H}_0\{u(r, z)\} = \tilde{u}(k, z) = \int_0^\infty rJ_0(kr)u(r, z)\,dr,$$

where $J_0(kr)$ is the Bessel function of order 0. Hence, the radial part of the laplacian in cylindrical coordinates transforms as follows.

$$\mathcal{H}_0\left\{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right\} = -k^2\tilde{u}(k, z)$$

The partial derivative with respect to z transforms like so.

$$\mathcal{H}_0\left\{\frac{\partial^n u}{\partial z^n}\right\} = \frac{d^n\tilde{u}}{dz^n}$$

Take the zero-order Hankel transform of both sides of the PDE.

$$\mathcal{H}_0\left\{u_{rr} + \frac{1}{r}u_r + u_{zz}\right\} = \mathcal{H}_0\{0\}$$

The Hankel transform is a linear operator.

$$\mathcal{H}_0\left\{u_{rr} + \frac{1}{r}u_r\right\} + \mathcal{H}_0\{u_{zz}\} = 0$$

Use the relations above to transform the partial derivatives.

$$-k^2\tilde{u}(k, z) + \frac{d^2\tilde{u}}{dz^2} = 0$$

Move the term with $\tilde{u}$ to the other side.

$$\frac{d^2\tilde{u}}{dz^2} = k^2\tilde{u}$$

The PDE has thus been reduced to an ODE whose solution can be expressed in terms of exponentials.

$$\tilde{u}(k,z) = A(k)e^{kz} + B(k)e^{-kz}$$

In order for $\tilde{u}$ to remain bounded as $z \to \infty$, we require $A(k) = 0$.

$$\tilde{u}(k,z) = B(k)e^{-kz}$$

Change back to $u(r,z)$ now by taking the inverse Hankel transform of $\tilde{u}(k,z)$.

$$u(r,z) = \mathcal{H}_0^{-1}\{\tilde{u}(k,z)\}$$

It is defined as

$$\mathcal{H}_0^{-1}\{\tilde{u}(k,z)\} = \int_0^\infty kJ_0(kr)\tilde{u}(k,z)\,dk,$$

so we have

$$u(r,z) = \int_0^\infty kJ_0(kr)B(k)e^{-kz}\,dk.$$

Use the provided boundary conditions at $z = 0$ to determine $B(k)$.

$$\begin{aligned} \text{For } 0 < r \leq a: &\quad u(r,0) = u_0 \quad \to \quad \int_0^\infty kJ_0(kr)B(k)\,dk = u_0 \\ \text{For } a < r < \infty: &\quad \frac{\partial u}{\partial z}(r,0) = 0 \quad \to \quad \int_0^\infty kJ_0(kr)B(k)(-k)\,dk = 0 \end{aligned}$$

The solution to these dual integral equations is

$$B(k) = \frac{2u_0}{\pi}\frac{\sin ka}{k^2}.$$

Plugging this in to the formula for $u(r,z)$, we get

$$u(r,z) = \int_0^\infty kJ_0(kr)\frac{2u_0}{\pi}\frac{\sin ka}{k^2}e^{-kz}\,dk.$$

Therefore,

$$u(r,z) = \frac{2u_0}{\pi}\int_0^\infty J_0(kr)\frac{\sin ka}{k}e^{-kz}\,dk.$$