Exercise 48

Solve the problem of the electrified unit disk in the (x,t)-plane with center at the origin. The electric potential $u(r,z)$ is axisymmetric and satisfies the boundary-value problem

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + u_{zz} &= 0, \quad 0 < r < \infty,\ 0 < z < \infty, \\ u(r,0) &= u_0, \quad 0 \le r \le a, \\ \frac{\partial u}{\partial z} &= 0, \quad \text{on } z = 0 \text{ for } a < r < \infty, \\ u(r,z) &\to 0 \quad \text{as } z \to \infty \text{ for all } r, \end{aligned}$$

where u_0 is constant. Show that the solution is

$$u(r,z) = \frac{2u_0}{\pi}\int_0^\infty J_0(kr)\frac{\sin ak}{k}e^{-kz}\,dk.$$