Exercise 9

Solve Example 1.6.2 with the initial data

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le \frac{\ell}{2}, \\ \ell - x & \text{if } \frac{\ell}{2} \le x \le \ell. \end{cases}$$

Solution

The initial boundary value problem that needs to be solved is the following:

$$\begin{aligned} & u_t = \kappa u_{xx}, && 0 < x < \ell,\ t > 0 \\ & u(0,t) = 0, && t > 0 \\ & u(\ell,t) = 0, && t > 0 \\ & u(x,0) = f(x) = \begin{cases} x & 0 \le x \le \frac{\ell}{2} \\ \ell - x & \frac{\ell}{2} \le x \le \ell \end{cases}, && 0 < x < \ell. \end{aligned}$$

The PDE and the boundary conditions are linear and homogeneous, which means that the method of separation of variables can be applied. Assume a product solution of the form, $u(x,t) = X(x)T(t)$, and substitute it into the PDE and boundary conditions to obtain

$$X(x)T'(t) = \kappa X''(x)T(t) \quad \to \quad \frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = k \tag{1.9.1}$$

$$\begin{aligned} u(0,t) = 0 \quad &\to \quad X(0)T(t) = 0 \quad \to \quad X(0) = 0 \\ u(\ell,t) = 0 \quad &\to \quad X(\ell)T(t) = 0 \quad \to \quad X(\ell) = 0. \end{aligned}$$

The left side of equation (1.9.1) is a function of t, and the right side is a function of x. Therefore, both sides must be equal to a constant. Values of this constant and the corresponding functions that satisfy the boundary conditions are known as eigenvalues and eigenfunctions, respectively. We have to examine three special cases: the case where the eigenvalues are positive ($k = \mu^2$), the case where the eigenvalue is zero ($k = 0$), and the case where the eigenvalues are negative ($k = -\lambda^2$). The solution to the PDE will be a linear combination of all product solutions.

Case I: Consider the Positive Eigenvalues ($k = \mu^2$)

Solving the ordinary differential equation in (1.9.1) for $X(x)$ gives

$$X''(x) = \mu^2 X(x), \quad X(0) = 0,\ X(\ell) = 0.$$

$$\begin{aligned} & X(x) = C_1 \cosh \mu x + C_2 \sinh \mu x \\ & X(0) = C_1 \quad \to \quad C_1 = 0 \\ & X(\ell) = C_2 \sinh \mu\ell = 0 \quad \to \quad C_2 = 0 \\ & X(x) = 0 \end{aligned}$$

Positive values of k lead to the trivial solution, $X(x) = 0$. Therefore, there are no positive eigenvalues and no associated product solutions.

Case II: Consider the Zero Eigenvalue ($k = 0$)

Solving the ordinary differential equation in (1.9.1) for $X(x)$ gives

$$X''(x) = 0, \quad X(0) = 0,\ X(\ell) = 0.$$

$$\begin{aligned} &X(x) = C_1 x + C_2 \\ &X(0) = C_2 \quad \rightarrow \quad C_2 = 0 \\ &X(\ell) = C_1\ell = 0 \quad \rightarrow \quad C_1 = 0 \\ &X(x) = 0 \end{aligned}$$

$k = 0$ leads to the trivial solution, $X(x) = 0$. Therefore, zero is not an eigenvalue, and there's no product solution associated with it.

Case III: Consider the Negative Eigenvalues ($k = -\lambda^2$)

Solving the ordinary differential equation in (1.9.1) for $X(x)$ gives

$$X''(x) = -\lambda^2 X(x), \quad X(0) = 0,\ X(\ell) = 0.$$

$$\begin{aligned} &X(x) = C_1 \cos\lambda x + C_2 \sin\lambda x \\ &X(0) = C_1 \quad \rightarrow \quad C_1 = 0 \\ &X(\ell) = C_2 \sin\lambda\ell = 0 \\ &\qquad\qquad \sin\lambda\ell = 0 \quad \rightarrow \quad \lambda\ell = n\pi,\ n = 1, 2, \dots \\ &X(x) = C_2 \sin\lambda x \qquad\qquad \lambda_n = \frac{n\pi}{\ell},\ n = 1, 2, \dots \end{aligned}$$

The eigenvalues are $k = -\lambda_n^2 = -\left(\frac{n\pi}{\ell}\right)^2$, and the corresponding eigenfunctions are $X_n(x) = \sin\frac{n\pi x}{\ell}$. Solving the ordinary differential equation for $T(t)$, $T'(t) = -\kappa\lambda^2 T(t)$, gives $T(t) = Ae^{-\kappa\lambda^2 t}$. The product solutions associated with the negative eigenvalues are thus $u_n(x,t) = X_n(x)T_n(t) = B\sin(\lambda_n x)e^{-\kappa\lambda_n^2 t}$ for $n = 1, 2, \dots$.

According to the principle of superposition, the solution to the PDE is a linear combination of all product solutions:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\kappa\left(\frac{n\pi}{\ell}\right)^2 t} \sin\frac{n\pi x}{\ell}.$$

The coefficients, B_n, are determined from the initial condition,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} = f(x). \tag{1.9.2}$$

Multiplying both sides of (1.9.2) by $\sin\frac{m\pi x}{\ell}$ (m being a positive integer) gives

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \sin\frac{m\pi x}{\ell} = f(x)\sin\frac{m\pi x}{\ell}.$$

Integrating both sides with respect to x from 0 to ℓ gives

$$\int_0^\ell \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \sin\frac{m\pi x}{\ell}\,dx = \int_0^\ell f(x)\sin\frac{m\pi x}{\ell}\,dx$$

$$\sum_{n=1}^{\infty} B_n \underbrace{\int_0^{\ell} \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx}_{=\frac{\ell}{2}\delta_{nm}} = \int_0^{\ell} f(x)\sin\frac{m\pi x}{\ell}\,dx$$

It is thanks to the orthogonality of the trigonometric functions that most terms in the infinite series vanish upon integration. Only the $n = m$ term remains, and this is denoted by the Kronecker delta function,

$$\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}.$$

$$B_n\left(\frac{\ell}{2}\right) = \int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$B_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx.$$

Now we substitute the initial data given in the problem statement for $f(x)$ to evaluate B_n.

$$B_n = \frac{2}{\ell}\left[\int_0^{\frac{\ell}{2}} x\sin\frac{n\pi x}{\ell}\,dx + \int_{\frac{\ell}{2}}^{\ell}(\ell - x)\sin\frac{n\pi x}{\ell}\,dx\right]$$

$$B_n = \frac{2}{\ell}\left[\frac{\ell}{n^2\pi^2}\left(-\frac{1}{2}n\pi l\cos\frac{n\pi}{2} + \ell\sin\frac{n\pi}{2}\right) + \frac{\ell^2}{2n^2\pi^2}\left(n\pi\cos\frac{n\pi}{2} + 2\sin\frac{n\pi}{2}\right)\right]$$

$$B_n = \frac{2}{\ell}\left(\frac{2\ell^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$B_n = \frac{4\ell}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Therefore, the solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\frac{4\ell}{n^2\pi^2}\sin\frac{n\pi}{2}e^{-\kappa\left(\frac{n\pi}{\ell}\right)^2 t}\sin\frac{n\pi x}{\ell}.}$$