

Vector Identity (11)

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

Proof

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{A}) &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left[\left(\sum_{j=1}^3 \delta_j \frac{\partial}{\partial x_j} \right) \times \left(\sum_{k=1}^3 \delta_k A_k \right) \right] \\
&= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left[\sum_{j=1}^3 \sum_{k=1}^3 (\delta_j \times \delta_k) \frac{\partial A_k}{\partial x_j} \right] \\
&= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \times \left(\sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \delta_l \varepsilon_{jkl} \frac{\partial A_k}{\partial x_j} \right) \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 (\delta_i \times \delta_l) \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \delta_m \varepsilon_{ilm} \varepsilon_{jkl} \frac{\partial}{\partial x_i} \frac{\partial A_k}{\partial x_j} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \delta_m \varepsilon_{mil} \varepsilon_{jkl} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m (\delta_{mj} \delta_{ik} - \delta_{mk} \delta_{ij}) \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m \delta_{mj} \delta_{ik} \frac{\partial^2 A_k}{\partial x_i \partial x_j} - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{m=1}^3 \delta_m \delta_{mk} \delta_{ij} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \delta_{ik} \frac{\partial^2 A_k}{\partial x_i \partial x_j} - \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \delta_{ij} \frac{\partial^2 A_k}{\partial x_i \partial x_j} \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \frac{\partial^2 A_k}{\partial x_k \partial x_j} - \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \frac{\partial^2 A_k}{\partial x_j \partial x_j} \\
&= \sum_{j=1}^3 \sum_{k=1}^3 \delta_j \frac{\partial^2 A_k}{\partial x_j \partial x_k} - \sum_{j=1}^3 \sum_{k=1}^3 \delta_k \frac{\partial^2 A_k}{\partial x_j^2} \\
&= \sum_{j=1}^3 \delta_j \frac{\partial}{\partial x_j} \left(\sum_{k=1}^3 \frac{\partial A_k}{\partial x_k} \right) - \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2} \left(\sum_{k=1}^3 \delta_k A_k \right) \\
&= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}
\end{aligned}$$