

Vector Identity (5)

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$$

Proof

$$\begin{aligned}
 \nabla \cdot (f\mathbf{A}) &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left[f \left(\sum_{j=1}^3 \delta_j A_j \right) \right] \\
 &= \left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_{j=1}^3 \delta_j A_j f \right) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \cdot \delta_j) \frac{\partial}{\partial x_i} A_j f \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \cdot \delta_j) \left(\frac{\partial A_j}{\partial x_i} f + A_j \frac{\partial f}{\partial x_i} \right) \\
 &= \sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \cdot \delta_j) \frac{\partial A_j}{\partial x_i} f + \sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \cdot \delta_j) A_j \frac{\partial f}{\partial x_i} \\
 &= f \left[\sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \cdot \delta_j) \frac{\partial A_j}{\partial x_i} \right] + \sum_{i=1}^3 \sum_{j=1}^3 (\delta_i \cdot \delta_j) A_i \frac{\partial f}{\partial x_j} \\
 &= f \left[\left(\sum_{i=1}^3 \delta_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_{j=1}^3 \delta_j A_j \right) \right] + \left(\sum_{i=1}^3 \delta_i A_i \right) \cdot \left(\sum_{j=1}^3 \delta_j \frac{\partial f}{\partial x_j} \right) \\
 &= f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f
 \end{aligned}$$