Problem 1.14

Let $P_{ab}(t)$ be the probability of finding the particle in the range $(a < x < b)$, at time t.

(a) Show that

$$\frac{dP_{ab}}{dt} = J(a,t) - J(b,t),$$

where

$$J(x,t) \equiv \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right).$$

What are the units of $J(x,t)$? *Comment:* J is called the **probability current**, because it tells you the rate at which probability is "flowing" past the point x. If $P_{ab}(t)$ is increasing, then more probability is flowing into the region at one end than flows out at the other.

(b) Find the probability current for the wave function in Problem 1.9. (This is not a very pithy example, I'm afraid; we'll encounter more substantial ones in due course.)

Solution

According to Born's interpretation, the probability of finding the particle in the interval $a < x < b$ at time t is

$$\begin{aligned}P_{ab}(t) &= \int_a^b |\Psi(x,t)|^2\,dx \\ &= \int_a^b \Psi(x,t)\Psi^*(x,t)\,dx.\end{aligned}$$

Differentiate both sides with respect to t.

$$\begin{aligned}\frac{dP_{ab}}{dt} &= \frac{d}{dt}\int_a^b \Psi(x,t)\Psi^*(x,t)\,dx \\ &= \int_a^b \frac{\partial}{\partial t}[\Psi(x,t)\Psi^*(x,t)]\,dx \\ &= \int_a^b \left(\frac{\partial\Psi}{\partial t}\Psi^* + \Psi\frac{\partial\Psi^*}{\partial t}\right)dx\end{aligned}$$

The governing equation for the wave function $\Psi(x,t)$ is Schrödinger's equation.

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi(x,t)$$

Take the complex conjugate of both sides to get the corresponding equation for Ψ^*.

$$\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V(x,t)\Psi^*(x,t)$$

Substitute these two formulas into the one for dP_{ab}/dt.

$$\begin{aligned}\frac{dP_{ab}}{dt} &= \int_a^b \left[\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)\Psi^* + \Psi\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\right]dx \\ &= \int_a^b \left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{i}{\hbar}V\Psi^*\Psi} - \frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \cancel{\frac{i}{\hbar}V\Psi^*\Psi}\right)dx\end{aligned}$$

Factor $i\hbar/2m$ and then write the integrand as a derivative with respect to x.

$$\begin{aligned}
\frac{dP_{ab}}{dt} &= \frac{i\hbar}{2m}\int_a^b \left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)dx \\
&= \frac{i\hbar}{2m}\int_a^b \left[\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial^2\Psi}{\partial x^2}\right) - \left(\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}\right)\right]dx \\
&= \frac{i\hbar}{2m}\int_a^b \left[\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\Psi^*}{\partial x}\Psi\right)\right]dx \\
&= \frac{i\hbar}{2m}\int_a^b \frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx \\
&= \frac{i\hbar}{2m}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_a^b \\
&= \frac{i\hbar}{2m}\left[\Psi^*(b,t)\frac{\partial\Psi}{\partial x}(b,t) - \frac{\partial\Psi^*}{\partial x}(b,t)\Psi(b,t)\right] - \frac{i\hbar}{2m}\left[\Psi^*(a,t)\frac{\partial\Psi}{\partial x}(a,t) - \frac{\partial\Psi^*}{\partial x}(a,t)\Psi(a,t)\right] \\
&= J(b,t) - J(a,t)
\end{aligned}$$

Each term in the equation,

$$\frac{dP_{ab}}{dt} = J(b,t) - J(a,t),$$

has to have the same units. Since P_{ab} is dimensionless, the SI units of J are inverse seconds (s^{-1}). The normalized wave function in Problem 1.9 is

$$\Psi(x,t) = \sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)+it]},$$

which means the probability current is

$$\begin{aligned}
J(x,t) &= \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) \\
&= \frac{i\hbar}{2m}\left[\left(\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)+it]}\right)\frac{\partial}{\partial x}\left(\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)-it]}\right)\right. \\
&\qquad\left. - \left(\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)-it]}\right)\frac{\partial}{\partial x}\left(\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)+it]}\right)\right] \\
&= \frac{i\hbar}{2m}\left[\left(\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)+it]}\right)\left(\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)-it]}\right)\left(-\frac{2amx}{\hbar}\right)\right. \\
&\qquad\left. - \left(\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)-it]}\right)\left(\sqrt[4]{\frac{2am}{\pi\hbar}}e^{-a[(mx^2/\hbar)+it]}\right)\left(-\frac{2amx}{\hbar}\right)\right] \\
&= \frac{i\hbar}{2m}\left[\sqrt{\frac{2am}{\pi\hbar}}e^{-2amx^2/\hbar}\left(-\frac{2amx}{\hbar}\right)\right. \\
&\qquad\left. - \sqrt{\frac{2am}{\pi\hbar}}e^{-2amx^2/\hbar}\left(-\frac{2amx}{\hbar}\right)\right] \\
&= 0.
\end{aligned}$$