Problem 1.16

A particle is represented (at time $t = 0$) by the wave function

$$\Psi(x,0) = \begin{cases} A(a^2 - x^2), & -a \leq x \leq +a, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine the normalization constant A.

(b) What is the expectation value of x?

(c) What is the expectation value of p? (Note that you *cannot* get it from $\langle p \rangle = md\langle x \rangle/dt$. Why not?)

(d) Find the expectation value of x^2.

(e) Find the expectation value of p^2.

(f) Find the uncertainty in x (σ_x).

(g) Find the uncertainty in p (σ_p).

(h) Check that your results are consistent with the uncertainty principle.

Solution

Normalize the wave function by requiring the integral of $|\Psi(x,0)|^2$ over the whole line to be 1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Psi(x,0)|^2 \, dx \\
&= \int_{-\infty}^{\infty} \Psi(x,0)\Psi^*(x,0) \, dx \\
&= \int_{-\infty}^{-a} 0^2 \, dx + \int_{-a}^{a} [A(a^2 - x^2)]^2 \, dx + \int_{a}^{\infty} 0^2 \, dx \\
&= \int_{-a}^{a} A^2(a^4 - 2a^2x^2 + x^4) \, dx \\
&= 2A^2 \int_0^a (a^4 - 2a^2x^2 + x^4) \, dx \\
&= 2A^2 \left(a^4 x - \frac{2a^2}{3}x^3 + \frac{x^5}{5} \right) \Bigg|_0^a \\
&= 2A^2 \left(a^5 - \frac{2a^5}{3} + \frac{a^5}{5} \right) \\
&= \frac{16a^5}{15} A^2
\end{aligned}$$

Solve for A.

$$A = \frac{1}{4}\sqrt{\frac{15}{a^5}}$$

According to Born's interpretation, $|\Psi(x,0)|^2$ represents the probability distribution for the particle's position at time $t=0$. Use it to calculate the expectation value of x at $t=0$.

$$\begin{aligned}\langle x\rangle = \frac{\int_{-\infty}^{\infty} x|\Psi(x,0)|^2\,dx}{\int_{-\infty}^{\infty}|\Psi(x,0)|^2\,dx} = \frac{\int_{-\infty}^{\infty} x|\Psi(x,0)|^2\,dx}{1} &= \int_{-\infty}^{\infty} x|\Psi(x,0)|^2\,dx \\ &= \int_{-a}^{a} x\left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2-x^2)\right]^2 dx \\ &= \frac{15}{16a^5}\int_{-a}^{a} x(a^2-x^2)^2\,dx \\ &= 0\end{aligned}$$

This last integral is zero because the integrand is an odd function and the integration interval is symmetric. Now calculate the expectation value of x^2 at $t=0$.

$$\begin{aligned}\langle x^2\rangle = \frac{\int_{-\infty}^{\infty} x^2|\Psi(x,0)|^2\,dx}{\int_{-\infty}^{\infty}|\Psi(x,0)|^2\,dx} = \frac{\int_{-\infty}^{\infty} x^2|\Psi(x,0)|^2\,dx}{1} &= \int_{-\infty}^{\infty} x^2|\Psi(x,0)|^2\,dx \\ &= \int_{-a}^{a} x^2\left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2-x^2)\right]^2 dx \\ &= \frac{15}{16a^5}\int_{-a}^{a} x^2(a^2-x^2)^2\,dx \\ &= \frac{15}{8a^5}\int_{0}^{a} x^2(a^2-x^2)^2\,dx \\ &= \frac{15}{8a^5}\int_{0}^{a} (a^4x^2-2a^2x^4+x^6)\,dx \\ &= \frac{15}{8a^5}\left(a^4\times\frac{a^3}{3}-2a^2\times\frac{a^5}{5}+\frac{a^7}{7}\right) \\ &= \frac{a^2}{7}\end{aligned}$$

Then the uncertainty in x at $t=0$ is

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{7}} = \frac{a}{\sqrt{7}}.$$

Note that the wave function's time evolution is obtained by solving Schrödinger's equation.

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi(x,t)$$

Take the complex conjugate of both sides to get the corresponding equation for Ψ^*.

$$\frac{\partial\Psi^*}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V(x,t)\Psi^*(x,t)$$

Use Ehrenfest's theorem to calculate the expectation value of p at time t.

$$
\begin{aligned}
\langle p\rangle = m\langle v\rangle = m\frac{d\langle x\rangle}{dt} &= m\frac{d}{dt}\left[\frac{\int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx}{\int_{-\infty}^{\infty}|\Psi(x,t)|^2\,dx}\right] \\
&= m\frac{d}{dt}\left[\frac{\int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx}{1}\right] \\
&= m\frac{d}{dt}\int_{-\infty}^{\infty} x|\Psi(x,t)|^2\,dx \\
&= m\frac{d}{dt}\int_{-\infty}^{\infty} x\Psi(x,t)\Psi^*(x,t)\,dx \\
&= m\int_{-\infty}^{\infty}\frac{\partial}{\partial t}\left[x\Psi(x,t)\Psi^*(x,t)\right]dx \\
&= m\int_{-\infty}^{\infty} x\left(\frac{\partial\Psi}{\partial t}\Psi^* + \Psi\frac{\partial\Psi^*}{\partial t}\right)dx \\
&= m\int_{-\infty}^{\infty} x\left[\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right)\Psi^* + \Psi\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right)\right]dx \\
&= m\int_{-\infty}^{\infty} x\left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{i}{\hbar}V\Psi^*\Psi} - \frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \cancel{\frac{i}{\hbar}V\Psi^*\Psi}\right)dx \\
&= \frac{i\hbar}{2}\int_{-\infty}^{\infty} x\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)dx \\
&= \frac{i\hbar}{2}\int_{-\infty}^{\infty} x\left[\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial^2\Psi}{\partial x^2}\right) - \left(\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}\right)\right]dx \\
&= \frac{i\hbar}{2}\int_{-\infty}^{\infty} x\left[\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\Psi^*}{\partial x}\Psi\right)\right]dx \\
&= \frac{i\hbar}{2}\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx \\
&= \frac{i\hbar}{2}\Bigg[\underbrace{x\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty}\underbrace{\frac{\partial}{\partial x}(x)}_{=1}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx\Bigg] \\
&= -\frac{i\hbar}{2}\int_{-\infty}^{\infty}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx \\
&= -\frac{i\hbar}{2}\left(\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx - \int_{-\infty}^{\infty}\frac{\partial\Psi^*}{\partial x}\Psi\,dx\right) \\
&= -\frac{i\hbar}{2}\left[\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx - \left(\underbrace{\Psi^*\Psi\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx\right)\right] \\
&= -i\hbar\int_{-\infty}^{\infty}\Psi^*\frac{\partial\Psi}{\partial x}\,dx \\
&= \int_{-\infty}^{\infty}\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi\,dx
\end{aligned}
$$

Now that the differentiation with respect to t is done, plug in $t = 0$ to find the expectation value of p at $t = 0$.

$$\begin{aligned}
\langle p \rangle &= -i\hbar \int_{-\infty}^{\infty} \Psi^*(x,0) \frac{\partial \Psi}{\partial x}(x,0)\, dx \\
&= -i\hbar \int_{-a}^{a} \left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2 - x^2)\right]^* \frac{\partial}{\partial x}\left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2 - x^2)\right] dx \\
&= -i\hbar \int_{-a}^{a} \left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2 - x^2)\right]\left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(-2x)\right] dx \\
&= \frac{15i\hbar}{8a^5} \int_{-a}^{a} x(a^2 - x^2)\, dx \\
&= 0
\end{aligned}$$

This last integral is zero because the integrand is an odd function and the integration interval is symmetric. Now calculate the expectation value of p^2 at $t = 0$.

$$\begin{aligned}
\langle p^2 \rangle &= \int_{-\infty}^{\infty} \Psi^*(x,0) \left(-i\hbar \frac{\partial}{\partial x}\right)^2 \Psi(x,0)\, dx \\
&= \int_{-\infty}^{\infty} \Psi^*(x,0) \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \Psi(x,0)\, dx \\
&= -\hbar^2 \int_{-\infty}^{\infty} \Psi^*(x,0) \frac{\partial^2 \Psi}{\partial x^2}(x,0)\, dx \\
&= -\hbar^2 \int_{-a}^{a} \left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2 - x^2)\right]^* \frac{\partial^2}{\partial x^2}\left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2 - x^2)\right] dx \\
&= -\hbar^2 \int_{-a}^{a} \left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(a^2 - x^2)\right]\left[\frac{1}{4}\sqrt{\frac{15}{a^5}}(-2)\right] dx \\
&= \frac{15\hbar^2}{8a^5} \int_{-a}^{a} (a^2 - x^2)\, dx \\
&= \frac{15\hbar^2}{4a^5} \int_{0}^{a} (a^2 - x^2)\, dx \\
&= \frac{15\hbar^2}{4a^5} \left(a^3 - \frac{a^3}{3}\right) \\
&= \frac{5\hbar^2}{2a^2}
\end{aligned}$$

Then the uncertainty in p at $t = 0$ is

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{5\hbar^2}{2a^2}} = \frac{\hbar}{a}\sqrt{\frac{5}{2}}.$$

The product of σ_x and σ_p at $t = 0$ is

$$\sigma_x \sigma_p = \left(\frac{a}{\sqrt{7}}\right)\left(\frac{\hbar}{a}\sqrt{\frac{5}{2}}\right) = \hbar\sqrt{\frac{5}{14}} \approx 0.598\hbar,$$

which is consistent with Heisenberg's uncertainty principle ($\sigma_x \sigma_p \geq \hbar/2 = 0.5\hbar$).