Problem 1.3

Consider the **gaussian** distribution

$$\rho(x) = Ae^{-\lambda(x-a)^2},$$

where A, a, and λ are positive real constants. (The necessary integrals are inside the back cover.)

(a) Use Equation 1.16 to determine A.

(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and σ.

(c) Sketch the graph of $\rho(x)$.

Solution

The needed integrals from inside the back cover are given here.

$$\int_0^\infty x^{2n} e^{-x^2/a^2}\,dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1} \tag{1}$$

$$\int_0^\infty x^{2n+1} e^{-x^2/a^2}\,dx = \frac{n!}{2}a^{2n+2} \tag{2}$$

Part (a)

The probability distribution here is gaussian, and it's assumed to be valid over the whole line.

$$\rho(x) = Ae^{-\lambda(x-a)^2}, \quad -\infty < x < \infty$$

Normalize the distribution by requiring the integral of $\rho(x)$ to be 1.

$$\int_{-\infty}^{\infty} \rho(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} Ae^{-\lambda(x-a)^2}\,dx = 1$$

Bring A out in front and make the substitution $u = x - a$. Then $du = dx$.

$$A\int_{-\infty}^{\infty} e^{-\lambda u^2}\,du = 1$$

$$A\int_{-\infty}^{\infty} e^{-u^2/(1/\sqrt{\lambda})^2}\,du = 1$$

Since the integrand is an even function of u, the integral can be taken from 0 to ∞ as long as a factor of 2 is placed in front.

$$2A\int_0^\infty e^{-u^2/(1/\sqrt{\lambda})^2}\,du = 1$$

Use equation (1) here with $n = 0$.

$$2A \cdot \sqrt{\pi}\left(\frac{\sqrt{\frac{1}{\lambda}}}{2}\right) = 1$$

Solve for A.

$$A = \sqrt{\frac{\lambda}{\pi}}$$

Therefore, the normalized probability distribution is

$$\rho(x) = \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2}, \quad -\infty < x < \infty.$$

Part (b)

Now calculate $\langle x \rangle$. Note that the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned}
\langle x \rangle = \frac{\int_{-\infty}^{\infty} x\rho(x)\,dx}{\int_{-\infty}^{\infty} \rho(x)\,dx} = \frac{\int_{-\infty}^{\infty} x\sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2}\,dx}{1} &= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (u+a) e^{-\lambda u^2}\,du \\
&= \sqrt{\frac{\lambda}{\pi}} \left(\underbrace{\int_{-\infty}^{\infty} u e^{-\lambda u^2}\,du}_{=0} + a \int_{-\infty}^{\infty} e^{-\lambda u^2}\,du \right) \\
&= \sqrt{\frac{\lambda}{\pi}} \left[2a \int_0^{\infty} e^{-u^2/(1/\sqrt{\lambda})^2}\,du \right] \\
&= \sqrt{\frac{\lambda}{\pi}} \left[2a \cdot \sqrt{\pi}\, \frac{\left(\frac{1}{\sqrt{\lambda}}\right)}{2} \right] \\
&= a
\end{aligned}$$

Then calculate $\langle x^2 \rangle$.

$$\begin{aligned}
\langle x^2 \rangle = \frac{\int_{-\infty}^{\infty} x^2\rho(x)\,dx}{\int_{-\infty}^{\infty} \rho(x)\,dx} &= \frac{\int_{-\infty}^{\infty} x^2\sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda(x-a)^2}\,dx}{1} \\
&= \sqrt{\frac{\lambda}{\pi}} \int_{-\infty}^{\infty} (u+a)^2 e^{-\lambda u^2}\,du \\
&= \sqrt{\frac{\lambda}{\pi}} \left(\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2}\,du + \underbrace{2a \int_{-\infty}^{\infty} u e^{-\lambda u^2}\,du}_{=0} + a^2 \int_{-\infty}^{\infty} e^{-\lambda u^2}\,du \right) \\
&= \sqrt{\frac{\lambda}{\pi}} \left[2 \int_0^{\infty} u^2 e^{-u^2/(1/\sqrt{\lambda})^2}\,du + 2a^2 \int_0^{\infty} e^{-u^2/(1/\sqrt{\lambda})^2}\,du \right] \\
&= \sqrt{\frac{\lambda}{\pi}} \left[2 \cdot \sqrt{\pi}\, \frac{(2)!}{1!} \left(\frac{\frac{1}{\sqrt{\lambda}}}{2} \right)^3 + 2a^2 \cdot \sqrt{\pi} \left(\frac{\frac{1}{\sqrt{\lambda}}}{2} \right) \right] \\
&= \sqrt{\frac{\lambda}{\pi}} \left(\frac{1}{2}\sqrt{\frac{\pi}{\lambda^3}} + a^2 \sqrt{\frac{\pi}{\lambda}} \right) \\
&= \frac{1}{2\lambda} + a^2
\end{aligned}$$

To determine the standard deviation, use equation (1.19) in the textbook.

$$\begin{aligned}\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \qquad (1.19)\\ &= \sqrt{\left(\frac{1}{2\lambda} + a^2\right) - (a)^2}\\ &= \sqrt{\frac{1}{2\lambda}}\\ &= \frac{1}{\sqrt{2\lambda}}\end{aligned}$$

Part (c)

Below is a plot of $\rho(x)$ versus x for the special case that $\lambda = 1$ and $a = 2$.