Problem 1.4

At time $t = 0$ a particle is represented by the wave function

$$\Psi(x,0) = \begin{cases} A(x/a), & 0 \le x \le a, \\ A(b-x)/(b-a), & a \le x \le b, \\ 0, & \text{otherwise,} \end{cases}$$

where A, a, and b are (positive) constants.

(a) Normalize Ψ (that is, find A, in terms of a and b).

(b) Sketch $\Psi(x,0)$, as a function of x.

(c) Where is the particle most likely to be found, at $t = 0$?

(d) What is the probability of finding the particle to the left of a? Check your result in the limiting cases $b = a$ and $b = 2a$.

(e) What is the expectation value of x?

Solution

Normalize the wave function by requiring the integral of $|\Psi(x,0)|^2$ over all x to be 1.

$$\int_{-\infty}^{\infty} |\Psi(x,0)|^2\,dx = 1$$

Split up the integral over the intervals that $\Psi(x,0)$ is defined on.

$$\int_{-\infty}^{0} |\Psi(x,0)|^2\,dx + \int_{0}^{a} |\Psi(x,0)|^2\,dx + \int_{a}^{b} |\Psi(x,0)|^2\,dx + \int_{b}^{\infty} |\Psi(x,0)|^2\,dx = 1$$

Substitute the appropriate formulas and then simplify.

$$\int_{-\infty}^{0} |0|^2\,dx + \int_{0}^{a} \left|A\left(\frac{x}{a}\right)\right|^2\,dx + \int_{a}^{b} \left|A\left(\frac{b-x}{b-a}\right)\right|^2\,dx + \int_{b}^{\infty} |0|^2\,dx = 1$$

$$\int_{0}^{a} A^2\left(\frac{x^2}{a^2}\right)dx + \int_{a}^{b} A^2\frac{(b-x)^2}{(b-a)^2}\,dx = 1$$

$$\frac{A^2}{a^2}\int_{0}^{a} x^2\,dx + \frac{A^2}{(b-a)^2}\int_{a}^{b}(b-x)^2\,dx = 1$$

$$\frac{A^2}{a^2}\cdot\frac{a^3}{3} + \frac{A^2}{(b-a)^2}\cdot\frac{(b-a)^3}{3} = 1$$

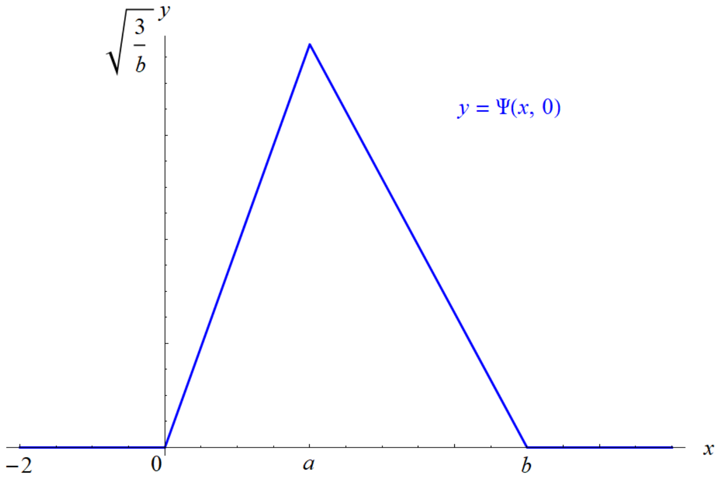
$$\frac{A^2}{3}\left[a + (b-a)\right] = 1$$

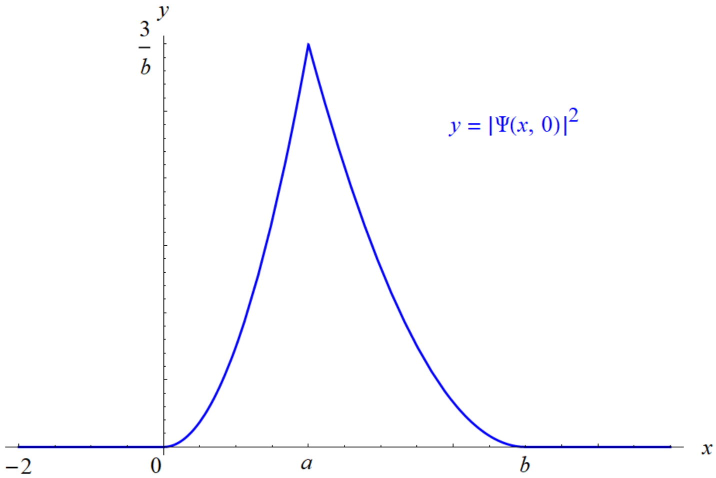
$$A = \sqrt{\frac{3}{b}}$$

With this value of A, the wave function becomes

$$\Psi(x,0) = \begin{cases} \sqrt{\frac{3}{b}}\frac{x}{a} & \text{if } 0 \leq x \leq a \\ \sqrt{\frac{3}{b}}\frac{b-x}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}.$$

Below are plots of $\Psi(x,0)$ versus x and $|\Psi(x,0)|^2$ versus x to illustrate the wave function's behavior initially.





According to the Born interpretation, it's $|\Psi(x,0)|^2$ that represents the probability distribution for the particle's position at $t = 0$. Even so, it can be seen from both graphs that $x = a$ is where the particle is most likely to be and that $x < 0$ and $x > b$ is where the particle cannot be. The probability of finding the particle to the left of a is

$$\begin{aligned} P &= \int_{-\infty}^{a} |\Psi(x,0)|^2 \, dx \\ &= \int_{-\infty}^{0} |\Psi(x,0)|^2 \, dx + \int_{0}^{a} |\Psi(x,0)|^2 \, dx \\ &= \int_{-\infty}^{0} |0|^2 \, dx + \int_{0}^{a} \left| \sqrt{\frac{3}{b}}\frac{x}{a} \right|^2 \, dx \\ &= \int_{0}^{a} \frac{3}{b}\frac{x^2}{a^2} \, dx \\ &= \frac{3}{a^2 b} \int_{0}^{a} x^2 \, dx \\ &= \frac{3}{a^2 b} \cdot \frac{a^3}{3} \\ &= \frac{a}{b}. \end{aligned}$$

If $b = a$, then $P = 1$. And if $b = 2a$, then $P = 1/2$.

Use the probability distribution to calculate the expectation value of x at $t = 0$.

$$\langle x \rangle = \frac{\int_{-\infty}^{\infty} x|\Psi(x,0)|^2\,dx}{\int_{-\infty}^{\infty} |\Psi(x,0)|^2\,dx} = \frac{\int_{-\infty}^{\infty} x|\Psi(x,0)|^2\,dx}{1} = \int_{-\infty}^{\infty} x|\Psi(x,0)|^2\,dx$$

$$= \int_0^a x\left|\sqrt{\frac{3}{b}}\left(\frac{x}{a}\right)\right|^2 dx + \int_a^b x\left|\sqrt{\frac{3}{b}}\left(\frac{b-x}{b-a}\right)\right|^2 dx$$

$$= \int_0^a x\cdot\frac{3}{b}\frac{x^2}{a^2}\,dx + \int_a^b x\cdot\frac{3}{b}\frac{(b-x)^2}{(b-a)^2}\,dx$$

$$= \frac{3}{a^2 b}\int_0^a x^3\,dx + \frac{3}{b(b-a)^2}\int_a^b x(b-x)^2\,dx$$

$$= \frac{3}{a^2 b}\cdot\frac{a^4}{4} + \frac{3}{b(b-a)^2}\cdot\frac{1}{12}(b-a)^3(3a+b)$$

$$= \frac{3a^2}{4b} + \frac{(b-a)(3a+b)}{4b}$$

$$= \frac{3a^2}{4b} + \frac{b^2+2ab-3a^2}{4b}$$

$$= \frac{b^2+2ab}{4b}$$

$$= \frac{b+2a}{4}$$