Problem 1.8

Suppose you add a constant V_0 to the potential energy (by "constant" I mean independent of x as well as t). In *classical* mechanics this doesn't change anything, but what about *quantum* mechanics? Show that the wave function picks up a time-dependent phase factor: $\exp(-iV_0t/\hbar)$. What effect does this have on the expectation value of a dynamical variable?

Solution

The governing equation for the wave function $\Psi(x,t)$ is the Schrödinger equation.

$$\frac{\partial\Psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Psi(x,t) \tag{1}$$

Suppose that $\Phi(x,t)$ is the wave function corresponding to a potential energy of $V(x,t)+V_0$.

$$\begin{aligned}\frac{\partial\Phi}{\partial t} &= \frac{i\hbar}{2m}\frac{\partial^2\Phi}{\partial x^2} - \frac{i}{\hbar}[V(x,t)+V_0]\Phi(x,t) \\ &= \frac{i\hbar}{2m}\frac{\partial^2\Phi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Phi(x,t) - \frac{iV_0}{\hbar}\Phi(x,t)\end{aligned}$$

Bring this last term to the left side.

$$\frac{\partial\Phi}{\partial t} + \frac{iV_0}{\hbar}\Phi(x,t) = \frac{i\hbar}{2m}\frac{\partial^2\Phi}{\partial x^2} - \frac{i}{\hbar}V(x,t)\Phi(x,t)$$

This PDE is first-order with respect to t, so an integrating factor,

$$I = \exp\left(\int^t \frac{iV_0}{\hbar}\,ds\right) = e^{iV_0t/\hbar},$$

can be used to simplify it. Multiply both sides of the PDE by I.

$$e^{iV_0t/\hbar}\frac{\partial\Phi}{\partial t} + \frac{iV_0}{\hbar}e^{iV_0t/\hbar}\Phi(x,t) = \frac{i\hbar}{2m}e^{iV_0t/\hbar}\frac{\partial^2\Phi}{\partial x^2} - \frac{i}{\hbar}e^{iV_0t/\hbar}V(x,t)\Phi(x,t)$$

The left side can be written as $\partial/\partial t(I\Phi)$ by the product rule.

$$\frac{\partial}{\partial t}\left(e^{iV_0t/\hbar}\Phi\right) = \frac{i\hbar}{2m}e^{iV_0t/\hbar}\frac{\partial^2\Phi}{\partial x^2} - \frac{i}{\hbar}e^{iV_0t/\hbar}V(x,t)\Phi(x,t)$$

The exponential function of t can be brought inside the partial derivative with respect to x.

$$\frac{\partial}{\partial t}\left(e^{iV_0t/\hbar}\Phi\right) = \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\left(e^{iV_0t/\hbar}\Phi\right) - \frac{i}{\hbar}V(x,t)\left(e^{iV_0t/\hbar}\Phi\right)$$

Comparing this to equation (1) and assuming the solution is unique, we see that

$$\Psi(x,t) = e^{iV_0t/\hbar}\Phi(x,t).$$

Therefore,

$$\Phi(x,t) = \Psi(x,t)e^{-iV_0t/\hbar}.$$

The expectation value for a dynamical variable of a particle with this new potential energy is

$$\begin{aligned}
\langle Q(x,p)\rangle &= \int_{-\infty}^{\infty} \Phi^*(x,t) Q\left(x, -i\hbar\frac{\partial}{\partial x}\right)\Phi(x,t)\,dx \\
&= \int_{-\infty}^{\infty} [\Psi(x,t)e^{-iV_0t/\hbar}]^* Q\left(x, -i\hbar\frac{\partial}{\partial x}\right)[\Psi(x,t)e^{-iV_0t/\hbar}]\,dx \\
&= \int_{-\infty}^{\infty} [\Psi^*(x,t)e^{iV_0t/\hbar}] Q\left(x, -i\hbar\frac{\partial}{\partial x}\right)[\Psi(x,t)e^{-iV_0t/\hbar}]\,dx \\
&= \int_{-\infty}^{\infty} \Psi^*(x,t) Q\left(x, -i\hbar\frac{\partial}{\partial x}\right)\Psi(x,t)\,dx,
\end{aligned}$$

which is the same as with the old potential energy.