Problem 2.10

(a) Construct $\psi_2(x)$.

(b) Sketch ψ_0, ψ_1, and ψ_2.

(c) Check the orthogonality of ψ_0, ψ_1, and ψ_2, by explicit integration. *Hint:* If you exploit the even-ness and odd-ness of the functions, there is really only one integral left to do.

Solution

Here we will solve the Schrödinger equation on the whole line with $V(x) = (1/2)m\omega^2x^2$,

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{2}m\omega^2x^2\Psi(x,t), \quad -\infty < x < \infty,$$

with the usual boundary conditions: $\Psi(x,t)$ and its derivatives tend to zero as $x \to \pm\infty$. Because the Schrödinger equation is linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $\Psi(x,t) = \psi(x)\phi(t)$ and plug it into the PDE.

$$i\hbar\frac{\partial}{\partial t}[\psi(x)\phi(t)] = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}[\psi(x)\phi(t)] + \frac{1}{2}m\omega^2x^2[\psi(x)\phi(t)]$$

Evaluate the derivatives.

$$i\hbar\psi(x)\phi'(t) = -\frac{\hbar^2}{2m}\psi''(x)\phi(t) + \frac{1}{2}m\omega^2x^2\psi(x)\phi(t)$$

Divide both sides by $\psi(x)\phi(t)$ to separate variables.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + \frac{1}{2}m\omega^2x^2$$

The only way a function of t can be equal to a function of x is if both are equal to a constant E.

$$i\hbar\frac{\phi'(t)}{\phi(t)} = -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + \frac{1}{2}m\omega^2x^2 = E$$

As a result of applying the method of separation of variables, Schrödinger's equation has reduced to two ODEs—one in each of the independent variables, x and t.

$$\left.\begin{aligned} i\hbar\frac{\phi'(t)}{\phi(t)} &= E \\ -\frac{\hbar^2}{2m}\frac{\psi''(x)}{\psi(x)} + \frac{1}{2}m\omega^2x^2 &= E \end{aligned}\right\}$$

Values of E for which the boundary conditions of this second equation are satisfied are known as eigenvalues (or eigenenergies in this context), and the nontrivial solutions $\psi(x)$ that satisfy this second equation are known as eigenfunctions (or eigenstates in this context). This ODE in x is known as the time-independent Schrödinger equation (TISE). Multiply both sides of it by $\psi(x)$.

$$-\frac{\hbar^2}{2m}\psi'' + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

Since $-\infty < x < \infty$, the method of operator factorization will be used to solve the TISE.

$$\frac{1}{2m}\left(-\hbar^2\frac{d^2\psi}{dx^2} + m^2\omega^2x^2\psi\right) = E\psi$$

$$\frac{1}{2m}\left[\left(-i\hbar\frac{d}{dx}\right)^2\psi + (m^2\omega^2x^2)\psi\right] = E\psi$$

$$\frac{1}{2m}\left(\hat{p}^2 + m^2\omega^2\hat{x}^2\right)\psi = E\psi \tag{1}$$

Because the position and momentum operators don't commute,

$$\begin{aligned}
[\hat{x},\hat{p}]f(x) &= (\hat{x}\hat{p} - \hat{p}\hat{x})f(x) \\
&= \hat{x}\hat{p}f(x) - \hat{p}\hat{x}f(x) \\
&= x\left(-i\hbar\frac{d}{dx}\right)f(x) - \left(-i\hbar\frac{d}{dx}\right)xf(x) \\
&= -i\hbar x\frac{df}{dx} + i\hbar\frac{d}{dx}[xf(x)] \\
&= \cancel{-i\hbar x\frac{df}{dx}} + i\hbar f(x) + \cancel{i\hbar x\frac{df}{dx}} \\
&= i\hbar f(x) \quad \Rightarrow \quad [\hat{x},\hat{p}] = i\hbar
\end{aligned}$$

there are two equivalent ways of writing the Hamiltonian operator in equation (1).

$$\begin{aligned}
\frac{1}{2m}\left(\hat{p}^2 + im\omega[\hat{x},\hat{p}] + m^2\omega^2\hat{x}^2 - im\omega[\hat{x},\hat{p}]\right)\psi &= E\psi \\
\frac{1}{2m}\left[\hat{p}^2 + im\omega(\hat{x}\hat{p} - \hat{p}\hat{x}) + m^2\omega^2\hat{x}^2 - im\omega(i\hbar)\right]\psi &= E\psi \\
\frac{1}{2m}\left(\hat{p}^2 + im\omega\hat{x}\hat{p} - im\omega\hat{p}\hat{x} + m^2\omega^2\hat{x}^2 + \hbar m\omega\right)\psi &= E\psi \\
\frac{1}{2m}\left[(-i\hat{p} + m\omega\hat{x})(i\hat{p} + m\omega\hat{x}) + \hbar m\omega\right]\psi &= E\psi \\
\hbar\omega\left(\frac{-i\hat{p} + m\omega\hat{x}}{\sqrt{2m\hbar\omega}}\frac{i\hat{p} + m\omega\hat{x}}{\sqrt{2m\hbar\omega}} + \frac{1}{2}\right)\psi &= E\psi
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2m}\left(\hat{p}^2 - im\omega[\hat{x},\hat{p}] + m^2\omega^2\hat{x}^2 + im\omega[\hat{x},\hat{p}]\right)\psi &= E\psi \\
\frac{1}{2m}\left[\hat{p}^2 - im\omega(\hat{x}\hat{p} - \hat{p}\hat{x}) + m^2\omega^2\hat{x}^2 + im\omega(i\hbar)\right]\psi &= E\psi \\
\frac{1}{2m}\left(\hat{p}^2 - im\omega\hat{x}\hat{p} + im\omega\hat{p}\hat{x} + m^2\omega^2\hat{x}^2 - \hbar m\omega\right)\psi &= E\psi \\
\frac{1}{2m}\left[(i\hat{p} + m\omega\hat{x})(-i\hat{p} + m\omega\hat{x}) - \hbar m\omega\right]\psi &= E\psi \\
\hbar\omega\left(\frac{i\hat{p} + m\omega\hat{x}}{\sqrt{2m\hbar\omega}}\frac{-i\hat{p} + m\omega\hat{x}}{\sqrt{2m\hbar\omega}} - \frac{1}{2}\right)\psi &= E\psi
\end{aligned}$$

Introducing the operators,

$$\hat{a}_+ = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x})$$
$$\hat{a}_- = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}),$$

the TISE can be written as

$$\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\psi = E\psi \quad \text{or} \quad \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\psi = E\psi.$$

According to Problem 2.3, E must be positive for there to be legitimate eigenstates. Expanding the left side of this first equation,

$$\hbar\omega\hat{a}_+(\hat{a}_-\psi) + \frac{\hbar\omega}{2}\psi = E\psi,$$

we see that the smallest positive energy is $E_0 = \hbar\omega/2$ and that the eigenstate associated with it (the ground state) satisfies

$$\hat{a}_-\psi_0 = 0.$$

Solve the resulting ODE for ψ_0.

$$\frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x})\psi_0 = 0$$
$$\frac{1}{\sqrt{2m\hbar\omega}}\left[i\left(-i\hbar\frac{d}{dx}\right) + m\omega x\right]\psi_0 = 0$$
$$\frac{1}{\sqrt{2m\hbar\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0 = 0$$
$$\hbar\frac{d\psi_0}{dx} + m\omega x\psi_0 = 0$$
$$\frac{\frac{d\psi_0}{dx}}{\psi_0} = -\frac{m\omega}{\hbar}x$$
$$\frac{d}{dx}\ln\psi_0 = -\frac{m\omega}{\hbar}x$$
$$\ln\psi_0 = -\frac{m\omega}{2\hbar}x^2 + C_0$$
$$\psi_0(x) = A_0\exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$

A_0 is a normalization constant, which is chosen so that the integral of $|\psi_0(x)|^2$ over the whole line is 1.

$$\begin{aligned}1 &= \int_{-\infty}^{\infty}|\psi_0(x)|^2\,dx\\ &= \int_{-\infty}^{\infty}A_0^2\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx\\ &= 2A_0^2\int_0^{\infty}\exp\left[-\frac{x^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right]dx = 2A_0^2\sqrt{\pi}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)\end{aligned}$$

Solve for A_0.

$$A_0^2\sqrt{\frac{\pi\hbar}{m\omega}} = 1$$

$$A_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

Therefore, the ground state of the harmonic oscillator is

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right).}$$

Since the left sides of the TISE are equivalent,

$$\hat{a}_+\hat{a}_- + \frac{1}{2} = \hat{a}_-\hat{a}_+ - \frac{1}{2}$$

$$\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1$$

$$[\hat{a}_-, \hat{a}_+] = 1.$$

$\hat{a}_+$ and $\hat{a}_-$ are known as the promotion and demotion operators, respectively, because they raise or lower a given eigenstate by 1 level. The energy of this new state can be calculated.

$$\begin{aligned}
\hat{H}(\hat{a}_+\psi) &= \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\hat{a}_+\psi \\
&= \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ + \frac{1}{2}\hat{a}_+\right)\psi \\
&= \hbar\omega\hat{a}_+\left(\hat{a}_-\hat{a}_+ + \frac{1}{2}\right)\psi \\
&= \hbar\omega\hat{a}_+\left(\hat{a}_+\hat{a}_- + 1 + \frac{1}{2}\right)\psi \\
&= \hat{a}_+\left[\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) + \hbar\omega\right]\psi \\
&= \hat{a}_+\left[\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\psi + \hbar\omega\psi\right] \\
&= \hat{a}_+\left(E\psi + \hbar\omega\psi\right) \\
&= \hat{a}_+\left(E + \hbar\omega\right)\psi \\
&= \left(E + \hbar\omega\right)\hat{a}_+\psi
\end{aligned}$$

$$\begin{aligned}
\hat{H}(\hat{a}_-\psi) &= \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\hat{a}_-\psi \\
&= \hbar\omega\left(\hat{a}_-\hat{a}_+\hat{a}_- - \frac{1}{2}\hat{a}_-\right)\psi \\
&= \hbar\omega\hat{a}_-\left(\hat{a}_+\hat{a}_- - \frac{1}{2}\right)\psi \\
&= \hbar\omega\hat{a}_-\left(\hat{a}_-\hat{a}_+ - 1 - \frac{1}{2}\right)\psi \\
&= \hat{a}_-\left[\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right) - \hbar\omega\right]\psi \\
&= \hat{a}_-\left[\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\psi - \hbar\omega\psi\right] \\
&= \hat{a}_-\left(E\psi - \hbar\omega\psi\right) \\
&= \hat{a}_-\left(E - \hbar\omega\right)\psi \\
&= \left(E - \hbar\omega\right)\hat{a}_-\psi
\end{aligned}$$

The energy of the first excited state is then

$$E_1 = \frac{\hbar\omega}{2} + \hbar\omega = \frac{3}{2}\hbar\omega,$$

and the eigenstate associated with it is

$$\begin{aligned}
\psi_1(x) &= A_1(\hat{a}_+\psi_0) \\
&= \frac{A_1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x})\psi_0(x) \\
&= \frac{A_1}{\sqrt{2m\hbar\omega}}\left[-i\left(-i\hbar\frac{d}{dx}\right) + m\omega x\right]\psi_0(x) \\
&= \frac{A_1}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d\psi_0}{dx} + m\omega x\psi_0\right) \\
&= \frac{A_1}{\sqrt{2m\hbar\omega}}\left\{-\hbar\frac{d}{dx}\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right] + m\omega x\left[\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]\right\} \\
&= \frac{A_1}{\sqrt{2m\hbar\omega}}\left[-\hbar\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(-\frac{m\omega}{\hbar}x\right) + m\omega x\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right] \\
&= \frac{A_1}{\sqrt{2m\hbar\omega}}\left[-\hbar\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\frac{m\omega}{\hbar}\right) + m\omega\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\right]x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= \frac{A_1}{\sqrt{2m\hbar\omega}}\left[2m\omega\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\right]x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= A_1\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= A_1\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right).
\end{aligned}$$

A_1 is a normalization constant, which is chosen so that the integral of $|\psi_1(x)|^2$ over the whole line is 1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}|\psi_1(x)|^2\,dx \\
&= \int_{-\infty}^{\infty}A_1^2\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2}x^2\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx \\
&= 2A_1^2\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2}\int_0^{\infty}x^2\exp\left[-\frac{x^2}{\left(\sqrt{\frac{\hbar}{m\omega}}\right)^2}\right]dx \\
&= 2A_1^2\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/2}\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{\sqrt{\frac{\hbar}{m\omega}}}{2}\right)^3 \\
&= A_1^2
\end{aligned}$$

Solve for A_1.

$$A_1 = 1$$

Therefore, the first excited state of the harmonic oscillator is

$$\boxed{\psi_1(x) = \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}x\exp\left(-\frac{m\omega}{2\hbar}x^2\right).}$$

The energy of the second excited state is

$$E_2 = \frac{3}{2}\hbar\omega + \hbar\omega = \frac{5}{2}\hbar\omega,$$

and the eigenstate associated with it is

$$\begin{aligned}
\psi_2(x) &= A_2(\hat{a}_+\psi_1) \\
&= \frac{A_2}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x})\psi_1(x) \\
&= \frac{A_2}{\sqrt{2m\hbar\omega}}\left[-i\left(-i\hbar\frac{d}{dx}\right) + m\omega x\right]\psi_1(x) \\
&= \frac{A_2}{\sqrt{2m\hbar\omega}}\left(-\hbar\frac{d\psi_1}{dx} + m\omega x\psi_1\right) \\
&= \frac{A_2}{\sqrt{2m\hbar\omega}}\left\{-\hbar\frac{d}{dx}\left[\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right] + m\omega x\left[\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right]\right\} \\
&= \frac{A_2}{\sqrt{2m\hbar\omega}}\left\{-\hbar\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}\left[\exp\left(-\frac{m\omega}{2\hbar}x^2\right) + x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(-\frac{m\omega}{\hbar}x\right)\right]\right. \\
&\qquad\left. + m\omega\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right\} \\
&= \frac{A_2}{\sqrt{2m\hbar\omega}}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}\left[-\hbar\exp\left(-\frac{m\omega}{2\hbar}x^2\right) + m\omega x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right) + m\omega x^2\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\right] \\
&= \frac{A_2}{\sqrt{2m\hbar\omega}}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}(2m\omega x^2 - \hbar)\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= A_2\sqrt{\frac{\hbar}{2m\omega}}\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
&= A_2\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right).
\end{aligned}$$

A_2 is a normalization constant, which is chosen so that the integral of $|\psi_2(x)|^2$ over the whole line is 1.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi_2(x)|^2\, dx \\
&= \int_{-\infty}^{\infty} A_2^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)^2\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx \\
&= 2A_2^2\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\int_0^{\infty}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)^2\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx
\end{aligned}$$

Make the following substitution.

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x$$

$$d\xi = \sqrt{\frac{m\omega}{\hbar}}\,dx \quad \rightarrow \quad dx = \sqrt{\frac{\hbar}{m\omega}}\,d\xi$$

Consequently,

$$
\begin{aligned}
1 &= 2A_2^2 \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int_0^\infty (2\xi^2 - 1)^2 e^{-\xi^2} \left(\sqrt{\frac{\hbar}{m\omega}}\, d\xi\right) \\
&= \frac{2A_2^2}{\pi^{1/2}} \int_0^\infty (2\xi^2 - 1)^2 e^{-\xi^2}\, d\xi \\
&= \frac{2A_2^2}{\pi^{1/2}} \int_0^\infty (4\xi^4 - 4\xi^2 + 1) e^{-\xi^2}\, d\xi \\
&= \frac{2A_2^2}{\pi^{1/2}} \left(4 \int_0^\infty \xi^4 e^{-\xi^2}\, d\xi - 4 \int_0^\infty \xi^2 e^{-\xi^2}\, d\xi + \int_0^\infty e^{-\xi^2}\, d\xi\right) \\
&= \frac{2A_2^2}{\pi^{1/2}} \left[4 \cdot \sqrt{\pi}\, \frac{4!}{2!} \left(\frac{1}{2}\right)^5 - 4 \cdot \sqrt{\pi}\, \frac{2!}{1!} \left(\frac{1}{2}\right)^3 + \sqrt{\pi} \left(\frac{1}{2}\right)\right] \\
&= 2A_2^2.
\end{aligned}
$$

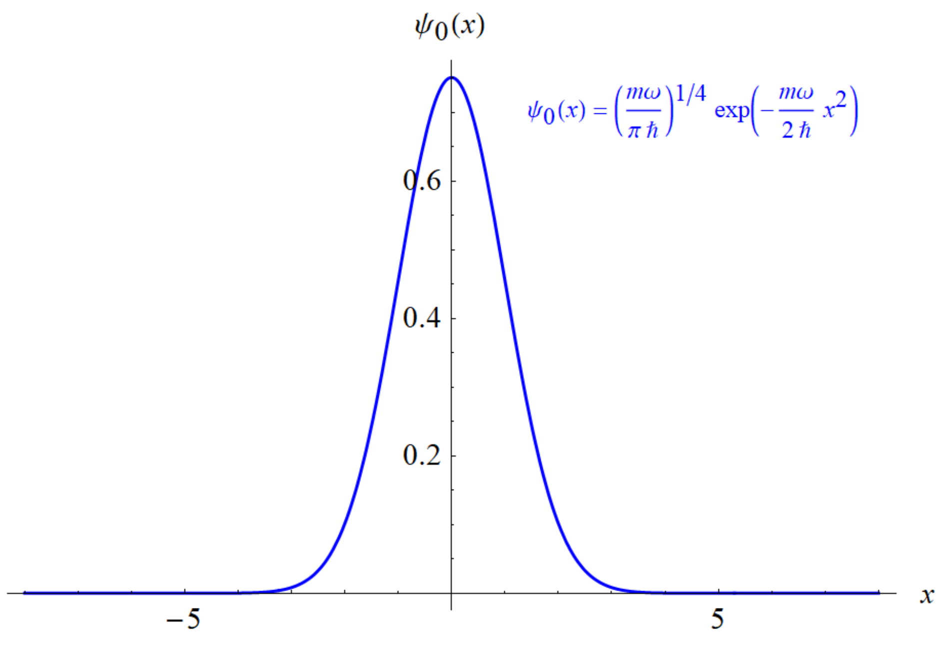
Solve for A_2.

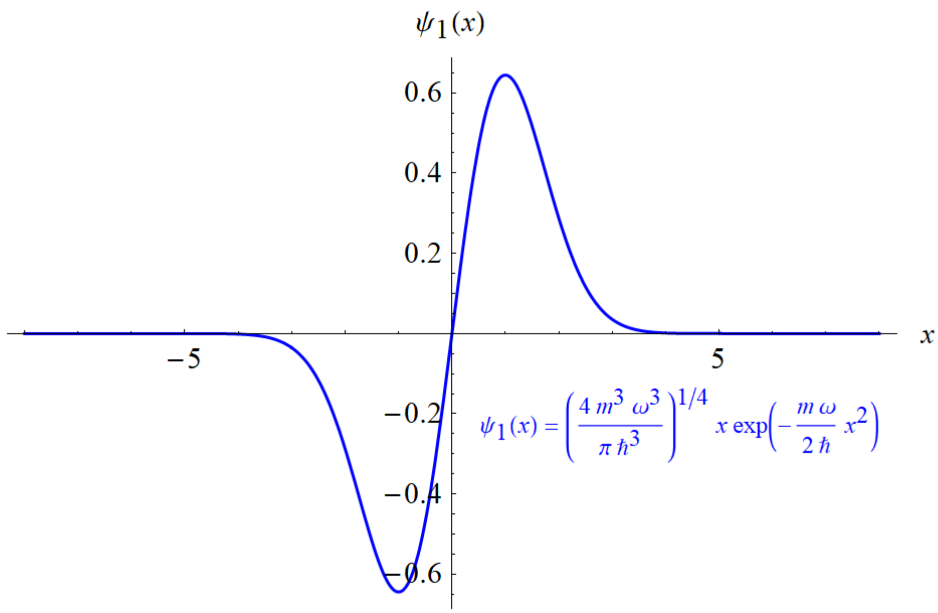
$$A_2 = \frac{1}{\sqrt{2}}$$

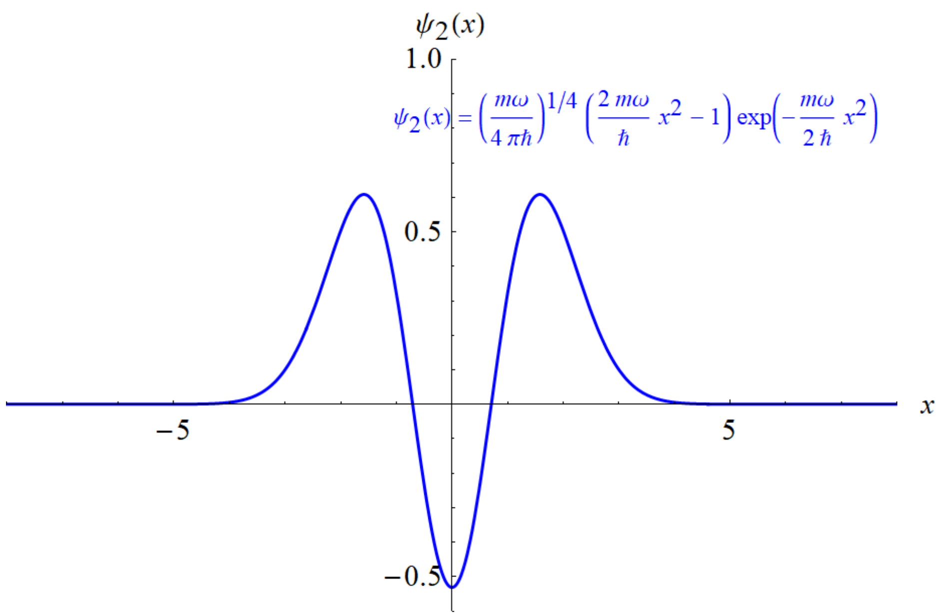
Therefore, the second excited state of the harmonic oscillator is

$$\boxed{\psi_2(x) = \left(\frac{m\omega}{4\pi\hbar}\right)^{1/4} \left(\frac{2m\omega}{\hbar} x^2 - 1\right) \exp\left(-\frac{m\omega}{2\hbar} x^2\right).}$$

We could keep using the promotion operator to get $\psi_3(x)$, $\psi_4(x)$, and so on if we wanted to. Below are plots of the first three eigenstates versus x for the special case that $m\omega/\hbar = 1$.







$\psi_0(x)$ and $\psi_1(x)$ are orthogonal because the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_0^*(x)\psi_1(x)\,dx &= \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x\exp\left(-\frac{m\omega}{2\hbar}x^2\right)dx \\ &= \sqrt{\frac{2}{\pi}}\frac{m\omega}{\hbar}\int_{-\infty}^{\infty} x\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx \\ &= 0\end{aligned}$$

$\psi_1(x)$ and $\psi_2(x)$ are orthogonal because the integral of an odd function over a symmetric interval is zero.

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_1^*(x)\psi_2(x)\,dx &= \int_{-\infty}^{\infty} \left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x \exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right)dx \\ &= \frac{1}{\sqrt{\pi}}\frac{m\omega}{\hbar}\int_{-\infty}^{\infty} x\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx \\ &= 0\end{aligned}$$

Now check the orthogonality of $\psi_0(x)$ and $\psi_2(x)$.

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_0^*(x)\psi_2(x)\,dx &= \int_{-\infty}^{\infty} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x^2\right)\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right)dx \\ &= \sqrt{\frac{m\omega}{2\pi\hbar}}\int_{-\infty}^{\infty}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx \\ &= 2\sqrt{\frac{m\omega}{2\pi\hbar}}\int_{0}^{\infty}\left(\frac{2m\omega}{\hbar}x^2 - 1\right)\exp\left(-\frac{m\omega}{\hbar}x^2\right)dx\end{aligned}$$

Make the following substitution.

$$\begin{aligned}\xi &= \sqrt{\frac{m\omega}{\hbar}}x \\ d\xi &= \sqrt{\frac{m\omega}{\hbar}}\,dx \quad \rightarrow \quad dx = \sqrt{\frac{\hbar}{m\omega}}\,d\xi\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_0^*(x)\psi_2(x)\,dx &= 2\sqrt{\frac{m\omega}{2\pi\hbar}}\int_0^{\infty}(2\xi^2 - 1)e^{-\xi^2}\left(\sqrt{\frac{\hbar}{m\omega}}\,d\xi\right) \\ &= \sqrt{\frac{2}{\pi}}\int_0^{\infty}(2\xi^2 - 1)e^{-\xi^2}\,d\xi \\ &= \sqrt{\frac{2}{\pi}}\left(2\int_0^{\infty}\xi^2 e^{-\xi^2}\,d\xi - \int_0^{\infty}e^{-\xi^2}\,d\xi\right) \\ &= \sqrt{\frac{2}{\pi}}\left[2\cdot\sqrt{\pi}\,\frac{2!}{1!}\left(\frac{1}{2}\right)^3 - \sqrt{\pi}\left(\frac{1}{2}\right)\right] \\ &= \sqrt{\frac{2}{\pi}}\left(\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\right) \\ &= 0,\end{aligned}$$

which means $\psi_0(x)$ and $\psi_2(x)$ are orthogonal as well. With the eigenenergies of the first three eigenstates known, the ODE in t can be solved to determine the corresponding $\phi(t)$ in each case.

$$i\hbar\frac{\phi_0'(t)}{\phi_0(t)} = \frac{\hbar\omega}{2} \qquad i\hbar\frac{\phi_1'(t)}{\phi_1(t)} = \frac{3}{2}\hbar\omega \qquad i\hbar\frac{\phi_2'(t)}{\phi_2(t)} = \frac{5}{2}\hbar\omega$$

$$\phi_0(t) = D_0 e^{-i\omega t/2} \qquad \phi_1(t) = D_1 e^{-3i\omega t/2} \qquad \phi_2(t) = D_2 e^{-5i\omega t/2}$$

According to the principle of superposition, the general solution to Schrödinger's equation is a linear combination of the product solutions (or stationary states in this context) $\Psi_n(x,t) = \psi_n(x)\phi_n(t)$ over all the eigenvalues.

$$\begin{aligned}
\Psi(x,t) &= \sum_{n=0}^{\infty} B_n \Psi_n(x,t) \\
&= B_0\Psi_0(x,t) + B_1\Psi_1(x,t) + B_2\Psi_2(x,t) + \cdots \\
&= B_0\psi_0(x)\phi_0(t) + B_1\psi_1(x)\phi_1(t) + B_2\psi_2(x)\phi_2(t) + \cdots \\
&= B_0\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) e^{-i\omega t/2} \\
&\qquad + B_1\left(\frac{4m^3\omega^3}{\pi\hbar^3}\right)^{1/4} x \exp\left(-\frac{m\omega}{2\hbar}x^2\right) e^{-3i\omega t/2} \\
&\qquad\qquad + B_2\left(\frac{m\omega}{4\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2 - 1\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right) e^{-5i\omega t/2} + \cdots
\end{aligned}$$

With a provided initial condition $\Psi(x,0)$ for the wave function, these coefficients B_n could be determined.