Problem 2.12

Find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, and $\langle T\rangle$, for the nth stationary state of the harmonic oscillator, using the method of Example 2.5. Check that the uncertainty principle is satisfied.

Solution

The promotion and demotion operators for the harmonic oscillator are defined as

$$\hat{a}_+ = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x})$$
$$\hat{a}_- = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}).$$

Solve this system of equations for $\hat{x}$ and $\hat{p}$.

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)$$
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$$

Note that the energy of the nth eigenstate is

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

As a result, the time-independent Schrödinger equation (TISE) becomes

$$\begin{aligned}
\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\psi &= E\psi \\
\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\psi_n &= \left(n + \frac{1}{2}\right)\hbar\omega\psi_n \\
\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\psi_n &= \left(n + \frac{1}{2}\right)\psi_n \\
\hat{a}_+\hat{a}_-\psi_n + \cancel{\frac{1}{2}\psi_n} &= n\psi_n + \cancel{\frac{1}{2}\psi_n} \\
\hat{a}_+\hat{a}_-\psi_n &= n\psi_n
\end{aligned}$$

$$\begin{aligned}
\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\psi &= E\psi \\
\hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\psi_n &= \left(n + \frac{1}{2}\right)\hbar\omega\psi_n \\
\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)\psi_n &= \left(n + \frac{1}{2}\right)\psi_n \\
\hat{a}_-\hat{a}_+\psi_n - \frac{1}{2}\psi_n &= n\psi_n + \frac{1}{2}\psi_n \\
\hat{a}_-\hat{a}_+\psi_n &= (n+1)\psi_n.
\end{aligned}$$

Calculate the expectation value of x.

$$\begin{aligned}
\langle x\rangle &= \int_{-\infty}^{\infty} \psi_n^*(x)(\hat{x})\psi_n(x)\,dx \\
&= \int_{-\infty}^{\infty} \psi_n^*(x)\left[\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)\right]\psi_n(x)\,dx \\
&= \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty} \psi_n^*(x)[\hat{a}_+\psi_n(x) + \hat{a}_-\psi_n(x)]\,dx \\
&= \sqrt{\frac{\hbar}{2m\omega}}\int_{-\infty}^{\infty} \psi_n^*(x)[A_{n+1}\psi_{n+1}(x) + A_{n-1}\psi_{n-1}(x)]\,dx
\end{aligned}$$

$$\begin{aligned}\langle x \rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[A_{n+1}\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_{n+1}(x)\,dx}_{=0} + A_{n-1}\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_{n-1}(x)\,dx}_{=0}\right] \\ &= 0\end{aligned}$$

These last two integrals are zero because the eigenstates are orthonormal. Now calculate the expectation value of x^2.

$$\begin{aligned}\langle x^2 \rangle &= \int_{-\infty}^{\infty}\psi_n^*(x)(\hat{x})^2\psi_n(x)\,dx \\ &= \int_{-\infty}^{\infty}\psi_n^*(x)\left[\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-)\right]^2\psi_n(x)\,dx \\ &= \int_{-\infty}^{\infty}\psi_n^*(x)\left[\frac{\hbar}{2m\omega}(\hat{a}_+ + \hat{a}_-)^2\right]\psi_n(x)\,dx \\ &= \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*(x)(\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2)\psi_n(x)\,dx \\ &= \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*(x)[\hat{a}_+^2\psi_n(x) + \hat{a}_+\hat{a}_-\psi_n(x) + \hat{a}_-\hat{a}_+\psi_n(x) + \hat{a}_-^2\psi_n(x)]\,dx \\ &= \frac{\hbar}{2m\omega}\int_{-\infty}^{\infty}\psi_n^*(x)[A_{n+2}\psi_{n+2}(x) + n\psi_n(x) + (n+1)\psi_n(x) + A_{n-2}\psi_{n-2}(x)]\,dx \\ &= \frac{\hbar}{2m\omega}\left[A_{n+2}\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_{n+2}(x)\,dx}_{=0} + n\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_n(x)\,dx}_{=1}\right. \\ &\qquad\left. + (n+1)\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_n(x)\,dx}_{=1} + A_{n-2}\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_{n-2}(x)\,dx}_{=0}\right] \\ &= \frac{\hbar}{2m\omega}[n + (n+1)] \\ &= \frac{\hbar}{2m\omega}(2n+1)\end{aligned}$$

Calculate the expectation value of p.

$$\begin{aligned}\langle p \rangle &= \int_{-\infty}^{\infty}\psi_n^*(x)(\hat{p})\psi_n(x)\,dx \\ &= \int_{-\infty}^{\infty}\psi_n^*(x)\left[i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)\right]\psi_n(x)\,dx \\ &= i\sqrt{\frac{\hbar m\omega}{2}}\int_{-\infty}^{\infty}\psi_n^*(x)[\hat{a}_+\psi_n(x) - \hat{a}_-\psi_n(x)]\,dx \\ &= i\sqrt{\frac{\hbar m\omega}{2}}\int_{-\infty}^{\infty}\psi_n^*(x)[A_{n+1}\psi_{n+1}(x) - A_{n-1}\psi_{n-1}(x)]\,dx \\ &= i\sqrt{\frac{\hbar m\omega}{2}}\left[A_{n+1}\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_{n+1}(x)\,dx}_{=0} - A_{n-1}\underbrace{\int_{-\infty}^{\infty}\psi_n^*(x)\psi_{n-1}(x)\,dx}_{=0}\right] = 0\end{aligned}$$

Calculate the expectation value of p^2.

$$
\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty} \psi_n^*(x)(\hat{p})^2\psi_n(x)\,dx \\
&= \int_{-\infty}^{\infty} \psi_n^*(x)\left[i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)\right]^2 \psi_n(x)\,dx \\
&= \int_{-\infty}^{\infty} \psi_n^*(x)\left[-\frac{\hbar m\omega}{2}(\hat{a}_+ - \hat{a}_-)^2\right] \psi_n(x)\,dx \\
&= -\frac{\hbar m\omega}{2}\int_{-\infty}^{\infty} \psi_n^*(x)(\hat{a}_+^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ + \hat{a}_-^2)\psi_n(x)\,dx \\
&= -\frac{\hbar m\omega}{2}\int_{-\infty}^{\infty} \psi_n^*(x)\left[\hat{a}_+^2\psi_n(x) - \hat{a}_+\hat{a}_-\psi_n(x) - \hat{a}_-\hat{a}_+\psi_n(x) + \hat{a}_-^2\psi_n(x)\right]dx \\
&= -\frac{\hbar m\omega}{2}\int_{-\infty}^{\infty} \psi_n^*(x)\left[A_{n+2}\psi_{n+2}(x) - n\psi_n(x) - (n+1)\psi_n(x) + A_{n-2}\psi_{n-2}(x)\right]dx \\
&= -\frac{\hbar m\omega}{2}\Bigg[A_{n+2}\underbrace{\int_{-\infty}^{\infty} \psi_n^*(x)\psi_{n+2}(x)\,dx}_{=0} - n\underbrace{\int_{-\infty}^{\infty} \psi_n^*(x)\psi_n(x)\,dx}_{=1} \\
&\qquad - (n+1)\underbrace{\int_{-\infty}^{\infty} \psi_n^*(x)\psi_n(x)\,dx}_{=1} + A_{n-2}\underbrace{\int_{-\infty}^{\infty} \psi_n^*(x)\psi_{n-2}(x)\,dx}_{=0}\Bigg] \\
&= -\frac{\hbar m\omega}{2}[-n-(n+1)] \\
&= \frac{\hbar m\omega}{2}(2n+1)
\end{aligned}
$$

The standard deviation in x is

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}(2n+1)},$$

and the standard deviation in p is

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar m\omega}{2}(2n+1)}.$$

Taking the product of these two,

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}(2n+1)}\sqrt{\frac{\hbar m\omega}{2}(2n+1)} = \frac{\hbar}{2}(2n+1),$$

we see that Heisenberg's uncertainty principle ($\sigma_x\sigma_p \geq \hbar/2$) is satisfied for the nth eigenstate. Finally, the expectation values of the potential energy V and kinetic energy T are

$$
\begin{aligned}
\langle V\rangle &= \left\langle \frac{1}{2}m\omega^2x^2\right\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{1}{2}m\omega^2\left[\frac{\hbar}{2m\omega}(2n+1)\right] = \frac{\hbar\omega}{4}(2n+1) \\
\langle T\rangle &= \left\langle \frac{p^2}{2m}\right\rangle = \frac{1}{2m}\langle p^2\rangle = \frac{1}{2m}\left[\frac{\hbar m\omega}{2}(2n+1)\right] = \frac{\hbar\omega}{4}(2n+1).
\end{aligned}
$$

Note that $\langle V\rangle + \langle T\rangle = E_n$.